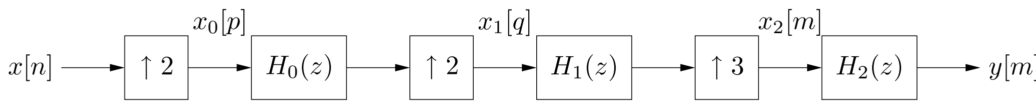
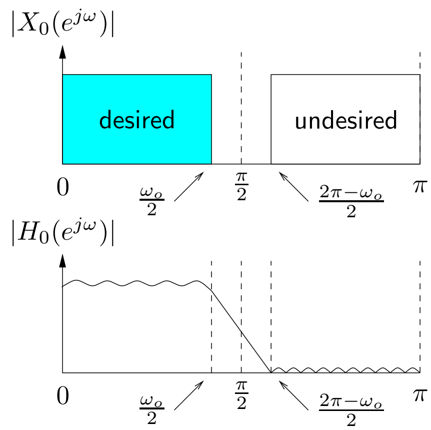
MIDTERM SOLUTIONS

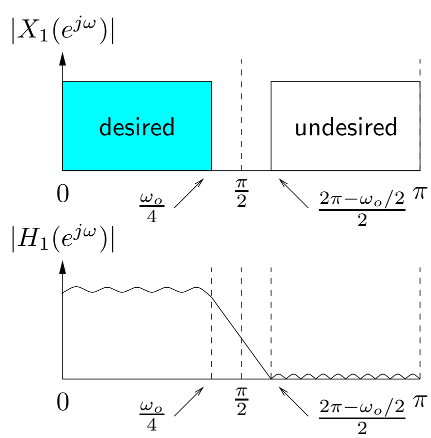
1. (a) We can consider the design of each filter based on the input signal bandwidth for that stage and the interpolation factor of that stage. Recall that $\{x[n]\}$ has bandwidth ω_o.



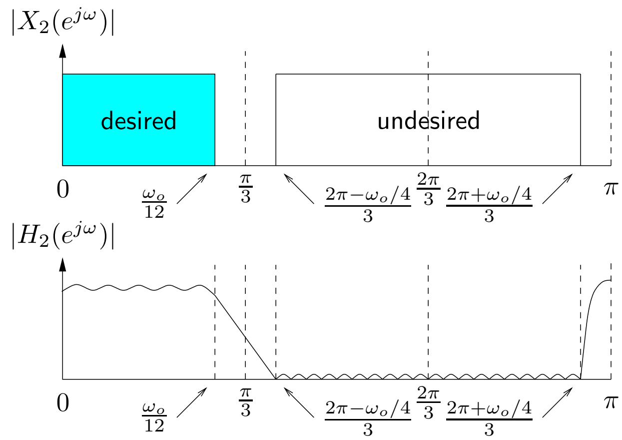
Due to interpolation of $x[n]$ by factor 2, the desired bandwidth of the input to $H_0(z)$ is $\frac{\omega_o}{2}$, giving passband cutoff at $\frac{\omega_o}{2}$ and stopband cutoff at $\frac{2\pi-\omega_o}{2}$. (See figure below).



Due to second-stage interpolation by factor 2, the desired bandwidth of the input to $H_1(z)$ is $\frac{\omega_o}{4}$, giving passband cutoff at $\frac{\omega_o}{4}$ and stopband cutoff at $\frac{2\pi-\omega_o/2}{2}$. (See figure below, which is unfortunately not drawn to scale!).



Due to third-stage interpolation by factor 3, the desired bandwidth of the input to $H_2(z)$ is $\frac{\omega_o}{12}$, giving passband cutoff at $\frac{\omega_o}{12}$ and first stopband cutoff at $\frac{2\pi-\omega_o/4}{3}$. The second stopband cutoff is at $\frac{2\pi+\omega_o/4}{3}$. (See figure below, which is unfortunately not drawn to scale!).



(b) The effective passband of the multistage filter is the cascade of the three single-stage passbands. Since passband ripples add when passbands are in cascade, we assign a passband ripple of $\frac{\delta_p}{3}$ to each:

$$\left(1+\frac{\delta_p}{3}\right)^3 \approx 1+\delta_p \text{ when } 0<\delta_p \ll 1.$$

The effective stopband of the multistage filter has regions where only one of the single-stage stopbands is cascaded with other passbands and/or transition bands. Assuming the gain of the transition bands is never more than 1, the stopbands must each be specified for ripple of δ_s:

$$\left(1+\frac{\delta_p}{3}\right)^2 \delta_s \approx \delta_s \text{ when } 0<\delta_p \ll 1.$$

To summarize:

	passband cutoff	stopband cutoff(s)	passband ripple	stopband ripple
$H_0(z)$	$\omega_o/2$	$\frac{2\pi-\omega_o}{2}$	$\delta_p/3$	δ_s
$H_1(z)$	$\omega_o/4$	$\frac{2\pi-\omega_o/2}{2}$	$\delta_p/3$	δ_s
$H_2(z)$	$\omega_o/12$	$\frac{2\pi-\omega_o/4}{3}$, $\frac{2\pi+\omega_o/4}{3}$	$\delta_p/3$	δ_s

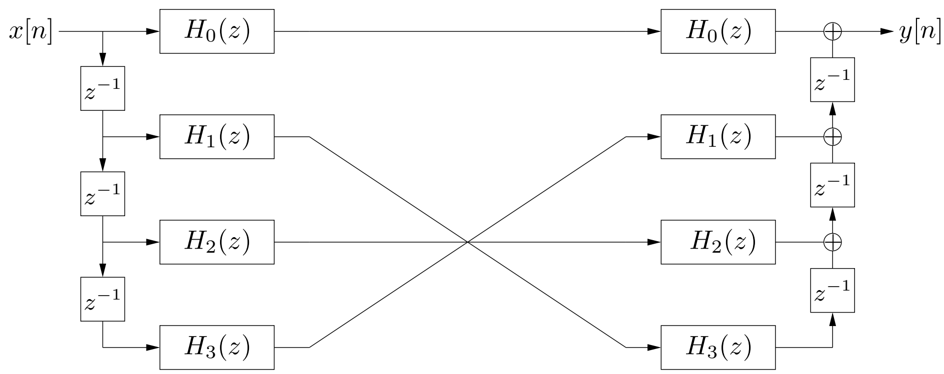
2. First we compute $\mathbf{W}_4\mathbf{W}_4$. Recalling that the k^{th} row and l^{th} column of $\mathbf{W}_N$ equals $e^{-j\frac{2\pi}{N}kl}$, we could say, for N-length DFTs, that

$$\begin{aligned}
[\mathbf{W}_N\mathbf{W}_N]_{pq} &= \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}pk}e^{-j\frac{2\pi}{N}kq} \\
&= \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(p+q)} \\
&= \begin{cases} N & \text{if } p+q \text{ is a multiple of } N \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

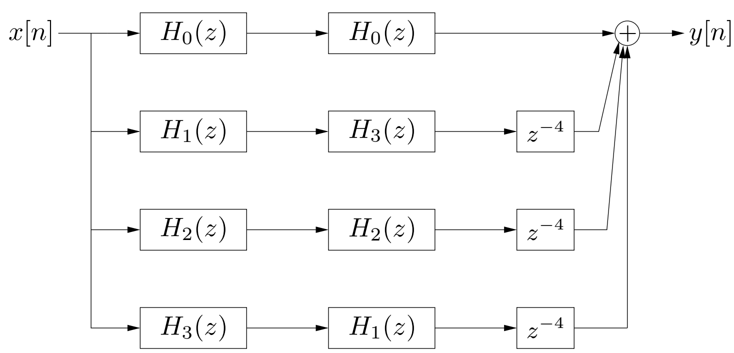
Alternatively, we could take the special case $N=4$ and manually compute the matrix product:

$$\mathbf{W}_4\mathbf{W}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

In either case, we could redraw the block diagram such that the first left output is connected to the first right input, the second left output is connected to the fourth right input, the third left output is connected to the third right input, and the fourth left output is connected to the second right input. The result is shown below.



Using the fact that all blocks are LTI, the block diagram could then be re-written as



Given that $H_k(e^{j\omega}) = \frac{1}{2}e^{jkd\omega}$ for $|\omega| < \pi$, we have (for $|\omega| < \pi$)

$$\begin{aligned} & & H_0(z)H_0(z)\big|_{z=e^{j\omega}} &= \frac{1}{4} \\ z^{-4}H_1(z)H_3(z)\big|_{z=e^{j\omega}} &= z^{-4}H_2(z)H_2(z)\big|_{z=e^{j\omega}} &&= \frac{1}{4}e^{-j\omega 4}e^{j\omega 4d} = \frac{1}{4}e^{j\omega 4(d-1)} \end{aligned}$$

so that the system output is

$$Y(e^{j\omega}) = \left(\frac{1}{4} + \frac{3}{4}e^{j\omega 4(d-1)}\right) X(e^{j\omega}) \text{ for } |\omega| < \pi.$$

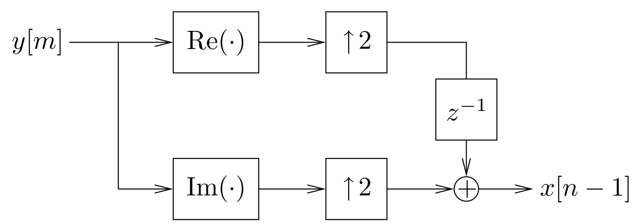
Thus, for perfect reconstruction, we choose $d = 1$. Note that this specifies the simple choice $H_k(z) = \frac{1}{2}z^k$, i.e., the k^{th} filter advances its input by k samples.

As a matter of fact, if you replace $H_k(z)$ with $\frac{1}{2}z^k$ and re-examine the block diagram in the problem statement, you will see that the filters cancel out the delays. This implies that the choice $\frac{1}{2}\mathbf{W}_4$ is not necessary for the matrix block; you can replace it with any matrix $\mathbf{A}$ such that the sum of the elements in $\mathbf{A}^2$ equals 4!

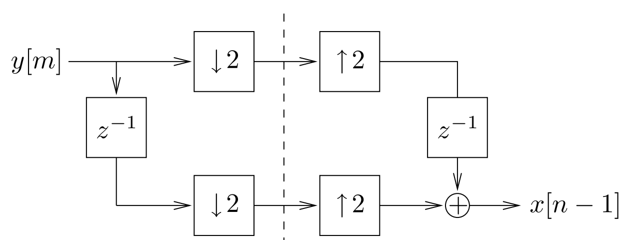
3. (a) We start our DTFT derivation in the z-domain, as usual.

$$\begin{aligned}
Y(z) &= \text{downsample-by-two}\{X(z)\} + j \cdot \text{downsample-by-two}\{X(z)z^{-1}\} \\
&= \frac{1}{2}\sum_{p=0}^{1} X(z^{\frac{1}{2}} e^{\pm j\frac{2\pi}{2}p}) + \frac{j}{2}\sum_{p=0}^{1} X(z^{\frac{1}{2}} e^{\pm j\frac{2\pi}{2}p}) \cdot (z^{\frac{1}{2}} e^{\pm j\frac{2\pi}{2}p})^{-1} \\
&= \frac{1}{2}\Big(X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\Big) + \frac{j}{2}\Big(X(z^{\frac{1}{2}})z^{-\frac{1}{2}} - X(-z^{\frac{1}{2}})z^{-\frac{1}{2}}\Big) \\
Y(e^{j\omega}) &= \frac{1}{2}\Big(X(e^{j\omega/2}) + X(e^{j(\omega/2\pm\pi)})\Big) + \frac{j}{2}\Big(X(e^{j\omega/2})e^{-j\omega/2} - X(e^{j(\omega/2\pm\pi)})e^{-j\omega/2}\Big) \\
&= \frac{1}{2}\Big(1 + je^{-j\omega/2}\Big)X(e^{j\omega/2}) + \frac{1}{2}\Big(1 - je^{-j\omega/2}\Big)X(e^{j(\omega/2\pm\pi)})
\end{aligned}$$

(b) Yes, the signal can be recovered by interleaving the real and imaginary components of $y[n]$ using the following structure.



To see why this is the case, realize that the overall system can be written as the block diagram below, where the top path carries the even samples $\{x[2n], n \in \mathbb{Z}\}$ and the bottom path carries the odd samples $\{x[2n+1], n \in \mathbb{Z}\}$.



Notice that both the even and odd components experience an input-output delay of one sample. If the path through the reconstruction delay went "up," the even samples would experience zero delay while the odd ones would experience a delay of two, causing improper interleaving at the output.