PRACTICE MIDTERM SOLUTIONS

1. There are two approaches to this problem. First, we have the frequency-domain approach:

$$\begin{aligned}
W(z) &= \frac{1}{D}\sum_{p=0}^{D-1} X(e^{\pm j\frac{2\pi}{D}p} z^{\frac{1}{D}}) \\
V(z) &= z^{-L}W(z) \;=\; z^{-L}\frac{1}{D}\sum_{p=0}^{D-1} X(e^{\pm j\frac{2\pi}{D}p} z^{\frac{1}{D}}) \\
Y(z) &= V(z^U) \;=\; z^{-LU}\frac{1}{D}\sum_{p=0}^{D-1} X(e^{\pm j\frac{2\pi}{D}p} z^{\frac{U}{D}}) \\
Y(e^{j\omega}) &= e^{-j\omega LU}\frac{1}{D}\sum_{p=0}^{D-1} X(e^{\pm j\frac{2\pi}{D}p} e^{j\omega\frac{U}{D}}) \\
&= e^{-j\omega LU}\frac{1}{D}\sum_{p=0}^{D-1} X(e^{j\frac{\omega U \pm 2\pi p}{D}}).
\end{aligned}$$

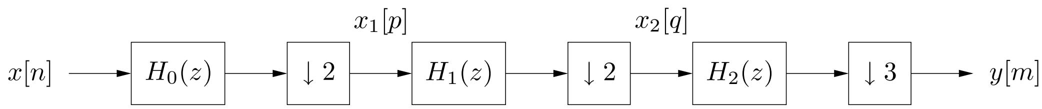
Next, we have the time-domain approach:

$$\begin{aligned}
w[l] &= x[lD] \\
v[l] &= w[l-L] \;=\; x[lD-LD] \\
y[m] &= \begin{cases} v[m/U] & \text{if } m \text{ is a multiple of } U \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} x[m\frac{D}{U}-LD] & \text{if } m \text{ is a multiple of } U \\ 0 & \text{else} \end{cases} \\
Y(e^{j\omega}) &= \sum_{m\,=\,\text{multiples of } U} x[m\frac{D}{U}-LD]e^{-j\omega m} \\
&= \sum_{l} x[lD-LD]e^{-j\omega lU} \\
&= \sum_{n} \underbrace{\left(\frac{1}{D}\sum_{p=0}^{D-1} e^{\pm j\frac{2\pi}{D}pn}\right)}_{=1 \text{ when } n \text{ is a multiple of } D,\text{ else } =0} x[n]e^{-j\omega(\frac{n}{D}+L)U},
\end{aligned}$$

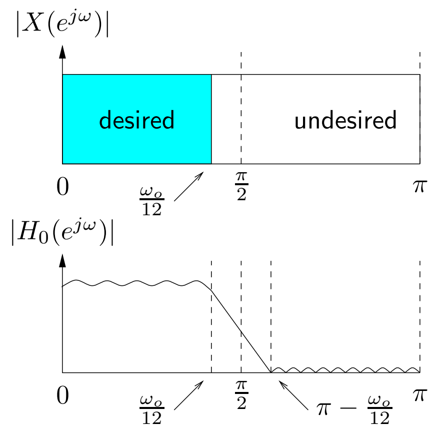
where we used the substitution $n = lD - LD$, i.e., $l = \frac{n}{D} + L$. Continuing,

$$\begin{aligned}
Y(e^{j\omega}) &= e^{-j\omega LU}\frac{1}{D}\sum_{p=0}^{D-1}\sum_{n} x[n]e^{-j\omega n\frac{U}{D}}e^{\pm j\frac{2\pi}{D}pn} \\
&= e^{-j\omega LU}\frac{1}{D}\sum_{p=0}^{D-1} X(e^{j\frac{\omega U \pm 2\pi p}{D}}).
\end{aligned}$$

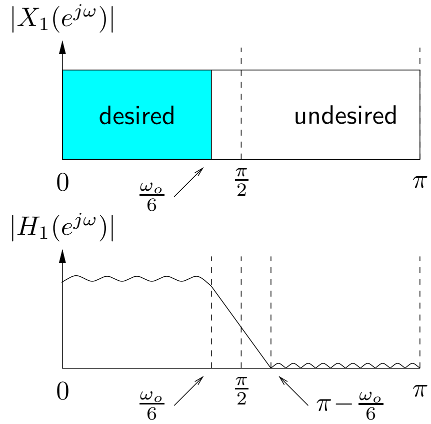
2. The design of each filter is based on the desired-signal-bandwidth at the input to that stage and the decimation factor of that stage. Recall that the desired-signal-bandwidth in $Y(e^{j\omega})$ is ω_o radians, and that the total decimation factor is $2 \times 2 \times 3 = 12$. Without loss of generality, we consider only positive frequencies below.



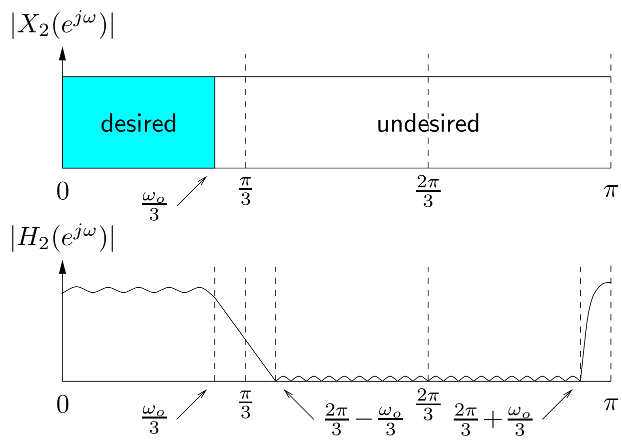
The signal $\{x[n]\}$ will be decimated by 12 to generate the output $\{y[n]\}$. Hence, the desired-signal-bandwidth of $X(e^{j\omega})$ is $\frac{\omega_o}{12}$, and the passband edge of $H_0(z)$ should be $\frac{\omega_o}{12}$. Since the first stage decimates by 2, we can choose the transition band to extend symmetrically around $\frac{\pi}{2}$ in order to prevent aliasing into the desired-signal region. Hence, the stopband edge should start at $\pi - \frac{\omega_o}{12}$. (See figure below, which is unfortunately not drawn to scale!).



The signal $\{x_1[p]\}$ will be decimated by 6 to generate the output $\{y[n]\}$. Hence, the desired-signal-bandwidth of $X_1(e^{j\omega})$ is $\frac{\omega_o}{6}$, and the passband edge of $H_1(z)$ should be $\frac{\omega_o}{6}$. Since the second stage decimates by 2, we can choose the transition band to extend symmetrically around $\frac{\pi}{2}$ in order to prevent aliasing into the desired-signal region. Hence, the stopband edge should start at $\pi - \frac{\omega_o}{6}$. (See figure below, which is unfortunately not drawn to scale!).



The signal $\{x_2[q]\}$ will be decimated by 3 to generate the output $\{y[n]\}$. Hence, the desired-signal-bandwidth of $X_2(e^{j\omega})$ is $\frac{\omega_o}{3}$, and the passband edge of $H_2(z)$ should be $\frac{\omega_o}{3}$. Since the last stage decimates by 3, we can choose the first transition band to extend symmetrically around $\frac{\pi}{3}$ in order to prevent aliasing into the desired-signal region. Hence, the first stopband edge should start at $\frac{2\pi}{3} - \frac{\omega_o}{6}$. But, after decimation, the region $[\frac{2\pi}{3} + \frac{\omega_o}{3}, \pi)$ will also avoid the desired-signal region, and so we can apply a stopband edge at $\frac{2\pi}{3} + \frac{\omega_o}{3}$. (See figure below.).



To summarize:

	passband edge	stopband edge(s)
$H_0(z)$	$\omega_o/12$	$\pi - \frac{\omega_o}{12}$
$H_1(z)$	$\omega_o/6$	$\pi - \frac{\omega_o}{6}$
$H_2(z)$	$\omega_o/3$	$\frac{2\pi}{3} - \frac{\omega_o}{3}$, $\frac{2\pi}{3} + \frac{\omega_o}{3}$

3. (a) If we define $\tilde{x}_c(t) := x_c(t+T+\Delta)$, then it is clear that

$$\begin{aligned}\tilde{X}_c(\Omega) &= \int_{-\infty}^{\infty} \tilde{x}_c(t)e^{-j\Omega t}dt \\ &= \int_{-\infty}^{\infty} x_c(t+T+\Delta)e^{-j\Omega t}dt \\ &= \int_{-\infty}^{\infty} x_c(\tau)e^{-j\Omega(\tau-T-\Delta)}dt \\ &= e^{j\Omega(T+\Delta)}X_c(\Omega)\end{aligned}$$

The top sampler generates $x_0[m]$, which is a sampling of $x_c(t)$ at rate $\frac{1}{2T}$, and the bottom sampler generates $x_1[m]$, which is a sampling of $\tilde{x}_c(t)$ at rate $\frac{1}{2T}$. Then

$$\begin{aligned}X_0(e^{j\omega}) &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega-2\pi k}{2T}\Big) \\ X_1(e^{j\omega}) &= \frac{1}{2T}\sum_k \tilde{X}_c\Big(\frac{\omega-2\pi k}{2T}\Big) \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega-2\pi k}{2T}\Big)e^{j(\frac{\omega-2\pi k}{2T})(T+\Delta)}\end{aligned}$$

Upsampling yields

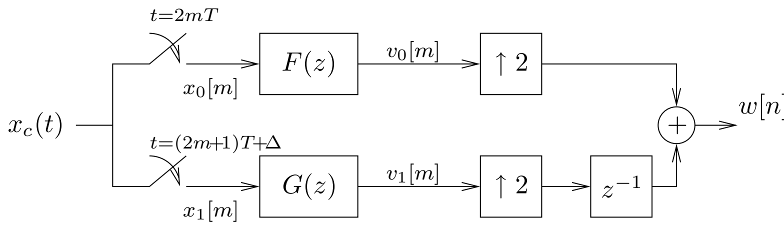
$$\begin{aligned}Y_0(e^{j\omega}) &= X_0(e^{j2\omega}) \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{2\omega-2\pi k}{2T}\Big) \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega-\pi k}{T}\Big) \\ Y_1(e^{j\omega}) &= e^{-j\omega}X_1(e^{j2\omega}) \\ &= e^{-j\omega}\frac{1}{2T}\sum_k X_c\Big(\frac{2\omega-2\pi k}{2T}\Big)e^{j(\frac{2\omega-2\pi k}{2T})(T+\Delta)} \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega-\pi k}{T}\Big)e^{j(\omega-\pi k)(1+\frac{\Delta}{T})-j\omega} \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega-\pi k}{T}\Big)e^{j(\omega-\pi k)\frac{\Delta}{T}}(-1)^k \\ Y(e^{j\omega}) &= Y_0(e^{j\omega})+Y_1(e^{j\omega}) \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega-\pi k}{T}\Big)\Big(1+e^{j(\omega-\pi k)\frac{\Delta}{T}}(-1)^k\Big)\end{aligned}$$

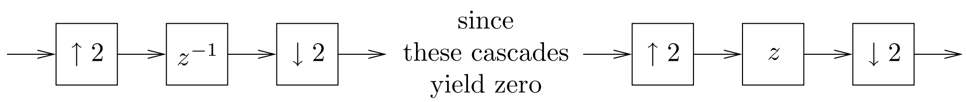
(b) When $\Delta = 0$, the previous expression reduces to

$$\begin{aligned}Y(e^{j\omega}) &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega-\pi k}{T}\Big)\Big(1+(-1)^k\Big) \\ &= \frac{1}{T}\sum_{k\text{ even}} X_c\Big(\frac{\omega-\pi k}{T}\Big) \\ &= \frac{1}{T}\sum_l X_c\Big(\frac{\omega-2\pi l}{T}\Big)\end{aligned}$$

which is the DTFT that corresponds to sampling $x_c(t)$ at rate $\frac{1}{T}$, i.e., $y[n] = x_c(nT)$.

(c) The block diagram can be simplified if we realize that the cascade of upsampling and down-sampling with a one-sample delay/advance in the middle yields zero.





(d) Now we consider the simplified block diagram with $F(z) = 1$ and $G(z)$ such that

$$G(e^{j\omega}) = e^{-j\omega\frac{\Delta}{2T}} \quad \text{for} \quad \omega \in [-\pi, \pi).$$

Then, for $\omega \in [-\pi, \pi)$:

$$\begin{aligned} V_1(e^{j\omega}) &= X_1(e^{j\omega})G(e^{j\omega}) \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega - 2\pi k}{2T}\Big)e^{j(\frac{\omega-2\pi k}{2T})(T+\Delta)}e^{-j\omega\frac{\Delta}{2T}} \quad \text{for} \quad \omega \in [-\pi, \pi) \end{aligned}$$

If this signal is upsampled by two and delayed by one sample, we get, for $\omega \in [-\pi, \pi)$:

$$\begin{aligned} &\frac{1}{2T}\sum_k X_c\Big(\frac{2\omega - 2\pi k}{2T}\Big)e^{j(\frac{2\omega-2\pi k}{2T})(T+\Delta)}e^{-j\omega\frac{\Delta}{T}}e^{-j\omega} \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega - \pi k}{T}\Big)e^{j(\omega-\pi k)(1+\frac{\Delta}{T})}e^{-j\omega(1+\frac{\Delta}{T})} \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega - \pi k}{T}\Big)e^{-j\pi k(1+\frac{\Delta}{T})} \end{aligned}$$

Since $F(z) = 1$, we have $V_0(e^{j\omega}) = X_0(e^{j\omega})$. Upsampling this by two gives

$$\frac{1}{2T}\sum_k X_c\Big(\frac{\omega - \pi k}{T}\Big).$$

Adding the last two signals to form $w[m]$, we get

$$\begin{aligned} W(e^{j\omega}) &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega - \pi k}{T}\Big) + \frac{1}{2T}\sum_k X_c\Big(\frac{\omega - \pi k}{T}\Big)e^{-j\pi k(1+\frac{\Delta}{T})} \\ &= \frac{1}{2T}\sum_k X_c\Big(\frac{\omega - \pi k}{T}\Big)\Big(1 + (-1)^k e^{-j\frac{\pi\Delta k}{T}}\Big) \\ &\neq \frac{1}{2T}\sum_k X_c\Big(\frac{\omega - \pi k}{T}\Big)\Big(1 + (-1)^k\Big) \\ &= \frac{1}{T}\sum_l X_c\Big(\frac{\omega - 2\pi l}{T}\Big) \end{aligned}$$

(unless Δ is a multiple of $2T$) and so

$$w[n] \neq w_c(nT).$$

In other words, the broken sampler cannot be fixed this way!