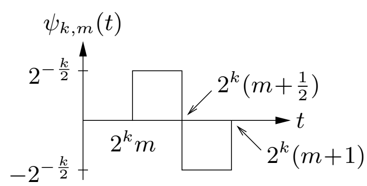
HOMEWORK SOLUTIONS #7

1. Since $\{\phi_{k,m}(t), k \in \mathbb{Z}, m \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{L}_2$, we can write

$$f(t) = \sum_{k,m} d_k[m]\psi_{k,m}(t)$$
$$d_k[m] = \langle \psi_{k,m}(t), f(t)\rangle = \int_{-\infty}^{\infty} \psi_{k,m}(t) f(t) dt$$

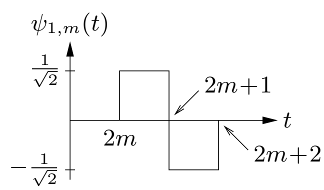
for any $f(t) \in \mathcal{L}_2$. The Haar wavelet is shown below at level k and shift m.



(a) When $f(t) = \phi(t)$, the coefficient expression reduces to

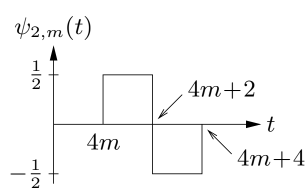
$$d_k[m] = \int_0^1 \psi_{k,m}(t)dt$$

- When $k \le 0$, we find that $\int_0^1 \psi_{k,m}(t)dt = 0$ for all shifts m since either $\psi_{k,m}(t)$ is zero inside of the region of integration or $\psi_{k,m}(t)$ has equal positive and negative contributions to the integral.
- When $k = 1$, we have



Thus $\psi_{1,m}(t) = 0$ under the integral unless $m = 0$, in which case $\int_0^1 \psi_{1,0}(t)dt = \frac{1}{\sqrt{2}}$.

- When $k = 2$, we have



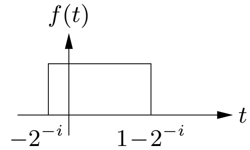
Thus $\psi_{1,m}(t) = 0$ under the integral unless $m = 0$, in which case $\int_0^1 \psi_{2,0}(t)dt = \frac{1}{2}$.

- Seeing a pattern emerge:

$$d_k[m] = \begin{cases} 2^{-\frac{k}{2}}\delta[m] & k \ge 1 \\ 0 & k \le 0 \end{cases}$$

(b) $\sum_{k,m} |\langle \psi_{k,m}(t), f(t)\rangle|^2 = \sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty} |2^{-\frac{k}{2}}\delta[m]|^2 = \sum_{k=1}^{\infty} 2^{-k} = 1.$

(c) This part follows the same reasoning as part (a), but uses the waveform $f(t) = \phi(t + 2^{-i})$ for $i \in \{1, 2, 3, \dots\}$, illustrated below.



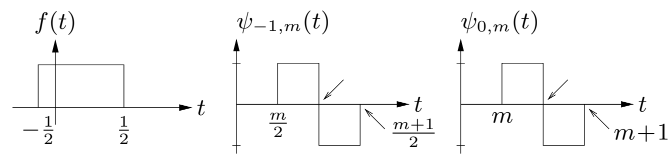
In this case

$$d_k[m] = \int_{-2^{-i}}^{1-2^{-i}} \psi_{k,m}(t)dt.$$

The main questions are: When is $\psi_{k,m}(t)$ non-zero within the region of integration? If $\psi_{k,m}(t)$ overlaps the region of integration, does it integrate to zero?

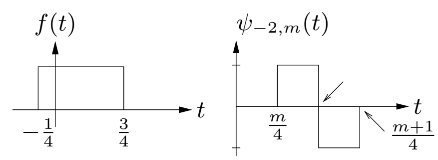
- $i = 1$:

k	m	$d_k[m]$
≤ -1	$\mathbb{Z}$	0
≥ 0	$\{-1, 0\}$	$\neq 0$
≥ 0	$\mathbb{Z} \setminus \{-1, 0\}$	0



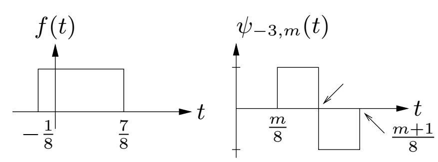
- $i = 2$:

k	m	$d_k[m]$
≤ -2	$\mathbb{Z}$	0
-1	$\{-1, 1\}$	$\neq 0$
-1	$\mathbb{Z} \setminus \{-1, 1\}$	0
≥ 0	$\{-1, 0\}$	$\neq 0$
≥ 0	$\mathbb{Z} \setminus \{-1, 0\}$	0



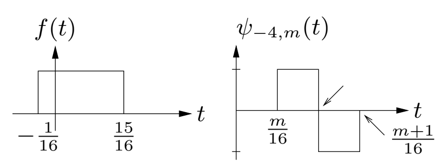
- $i = 3$:

k	m	$d_k[m]$
≤ -3	$\mathbb{Z}$	0
-2	$\{-1, 3\}$	$\neq 0$
-2	$\mathbb{Z} \setminus \{-1, 3\}$	0
-1	$\{-1, 1\}$	$\neq 0$
-1	$\mathbb{Z} \setminus \{-1, 1\}$	0
≥ 0	$\{-1, 0\}$	$\neq 0$
≥ 0	$\mathbb{Z} \setminus \{-1, 0\}$	0



- $i = 4$:

k	m	$d_k[m]$
≤ -4	$\mathbb{Z}$	0
-2	$\{-1, 7\}$	$\neq 0$
-2	$\mathbb{Z} \setminus \{-1, 7\}$	0
-2	$\{-1, 3\}$	$\neq 0$
-2	$\mathbb{Z} \setminus \{-1, 3\}$	0
-1	$\{-1, 1\}$	$\neq 0$
-1	$\mathbb{Z} \setminus \{-1, 1\}$	0
≥ 0	$\{-1, 0\}$	$\neq 0$
≥ 0	$\mathbb{Z} \setminus \{-1, 0\}$	0



Noticing a pattern, we claim

$$d_k[m] \neq 0 \text{ when } \begin{cases} m \in \{-1, 2^{-k} - 1\} & k \in \{0, -1, -2, ..., -i+1\} \\ m \in \{-1, 0\} & k > 0 \end{cases}$$

2. (a) We determined in class that, for $\{\phi(t-n), n \in \mathbb{Z}\}$ to be an orthonormal set, we need (for the real-valued case)

$$\delta[m] \;=\; \sum_n h[n]h[n-2m].$$

Evaluating the previous equation at $m = 0$,

$$1 \;=\; \sum_n |h[n]|^2.$$

(b) Using the scaling equation and the orthonormal property,

$$\begin{aligned}
\sqrt{2}\,\langle \phi(2t-m), \phi(t) \rangle &= \sqrt{2}\sqrt{2}\left\langle \phi(2t-m), \sum_n h[n]\phi(2t-n) \right\rangle \\
&= 2\sum_n h[n]\,\langle \phi(2t-m), \phi(2t-n) \rangle \\
&= 2\sum_n h[n] \int_{-\infty}^{\infty} \phi(2t-m)\phi(2t-n)dt \\
&= 2\sum_n h[n] \int_{-\infty}^{\infty} \phi(\tau-m)\phi(\tau-n)\frac{d\tau}{2} \\
&= \sum_n h[n]\delta[n-m] \\
&= h[m]
\end{aligned}$$

(c) Using the scaling equation,

$$\begin{aligned}
\Phi(\Omega) &= \int_{-\infty}^{\infty} \phi(t)e^{-j\Omega t}dt \\
&= \int_{-\infty}^{\infty} \sqrt{2}\sum_n h[n]\phi(2t-n)e^{-j\Omega t}dt \\
&= \sqrt{2}\sum_n h[n]e^{-j\frac{\Omega}{2}n} \int_{-\infty}^{\infty} \phi(2t-n)e^{-j\frac{\Omega}{2}(2t-n)}dt \\
&= \sqrt{2}H(e^{-j\frac{\Omega}{2}}) \int_{-\infty}^{\infty} \phi(\tau)e^{-j\frac{\Omega}{2}\tau}\frac{d\tau}{2} \\
&= \frac{1}{\sqrt{2}}H(e^{-j\frac{\Omega}{2}})\Phi(\tfrac{\Omega}{2})
\end{aligned}$$

3. (a) Here we leverage the result of problem 2(c).

$$\begin{aligned}
\sum_n h[n] &= \lim_{\Omega\to 0}\sum_n h[n]e^{-j\frac{\Omega}{2}} \\
&= \lim_{\Omega\to 0} H(e^{j\frac{\Omega}{2}}) \\
&= \sqrt{2}\lim_{\Omega\to 0}\frac{\Phi(\Omega)}{\Phi(\frac{\Omega}{2})} \\
&= \sqrt{2}
\end{aligned}$$

where we have assumed that $\Phi(0)$ is non-zero and that $\Phi(\Omega)$ is continuous at the origin.

(b) Realizing that $h[2n]$ are the even samples of $h[m]$ and that $h[2n+1]$ are the odd samples,

$$\begin{aligned}
\sum_n h[2n] - \sum_n h[2n+1] &= \sum_m h[m](-1)^m \\
&= H(z)\big|_{z=-1}
\end{aligned}$$

We know from part (a) that

$$H(z)\big|_{z=1} \;=\; \sum_n h[n] \;=\; \sqrt{2},$$

and we know that

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) \;=\; 2 \quad \forall z.$$

Evaluating the previous expression at z=1,

$$\begin{aligned} 2 &= H(1)^2 + H(-1)^2 \\ &= 2 + H(-1)^2 \\ \Rightarrow \quad 0 &= H(-1) \\ &= \sum_n h[2n] - \sum_n h[2n+1] \end{aligned}$$