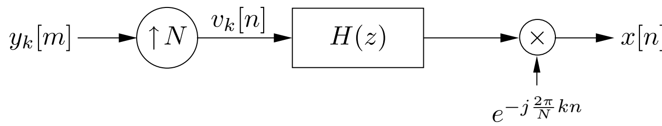
ECE-700 **Digital Signal Processing** **Winter 2007**

Homework #5 Feb. 7, 2007

HOMEWORK SOLUTIONS #5

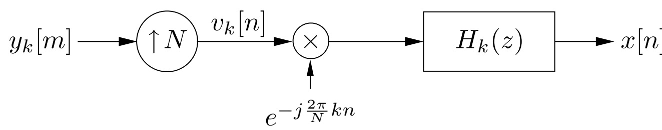
1. (a) Consider the k^{th} branch.



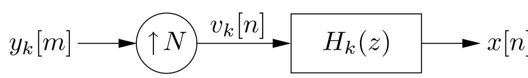
We have

$$\begin{aligned} x_k[n] &= \left(\sum_i h[i]v_k[n-i]\right)e^{-j\frac{2\pi}{N}kn} \\ &= \sum_i \underbrace{h[i]e^{-j\frac{2\pi}{N}ki}}_{h_k[i]} v_k[n-i]e^{-j\frac{2\pi}{N}k(n-i)} \end{aligned}$$

where we have rewritten the branch output as the convolution of a modulated filter response $h_k[i]$ with a modulated input $v_k[n]e^{-j\frac{2\pi}{N}kn}$.



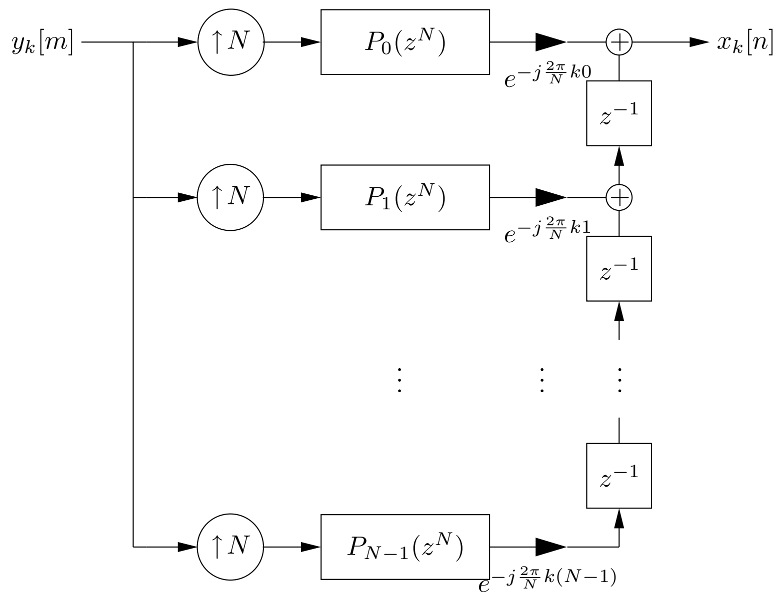
Noting that $v_k[n]$ is zero-valued unless n is a multiple of N, and that $e^{-j\frac{2\pi}{N}kn}$ equals one when n is a multiple of N, we see that the modulation has no effect. Hence the block diagram can be rewritten



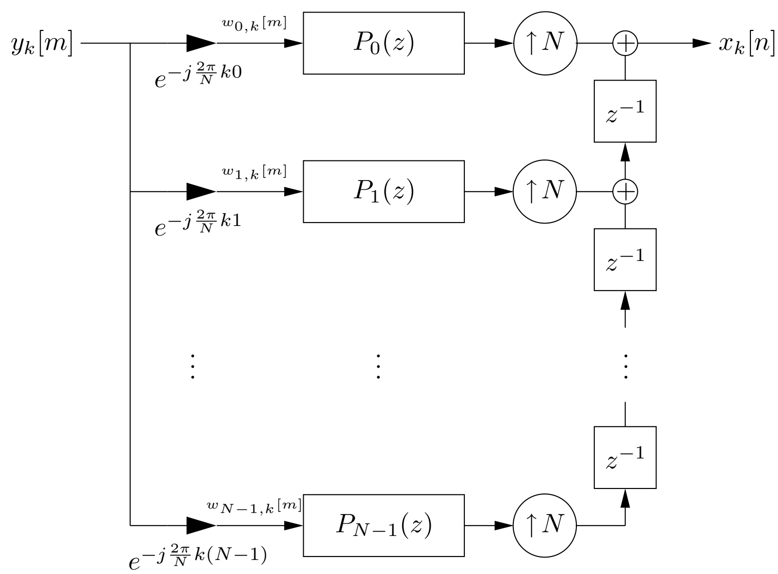
To apply the Noble identity, we must expand $H_k(z)$ into a parallel bank of upsampled polyphase filters. Defining $p_\ell[m] = h[mN+\ell]$, we find

$$\begin{aligned} h_k[mN+\ell] &= h[mN+\ell]e^{-j\frac{2\pi}{N}k(mN+\ell)} \\ &= p_\ell[m]e^{-j\frac{2\pi}{N}k\ell} \\ H_k(z) &= \sum_{i=-\infty}^{\infty} h_k[i]z^{-i} \\ &= \sum_{m=-\infty}^{\infty}\sum_{\ell=0}^{N-1} h_k[mN+\ell]z^{-mN+\ell} \\ &= \sum_{\ell=0}^{N-1} z^{-\ell}e^{-j\frac{2\pi}{N}k\ell} \sum_{m=-\infty}^{\infty} p_\ell[m](z^N)^{-m} \\ &= \sum_{\ell=0}^{N-1} P_\ell(z^N)z^{-\ell}e^{-j\frac{2\pi}{N}k\ell} \end{aligned}$$

thus the block diagram for the k^{th} filterbank branch becomes



Realizing that the gain $e^{-j\frac{2\pi}{N}k\ell}$ on the ℓ^{th} polyphase branch is not a function of the time index, we can put it before the filter and upsampler. Combining this with the use of a Noble identity, we get the equivalent diagram below.



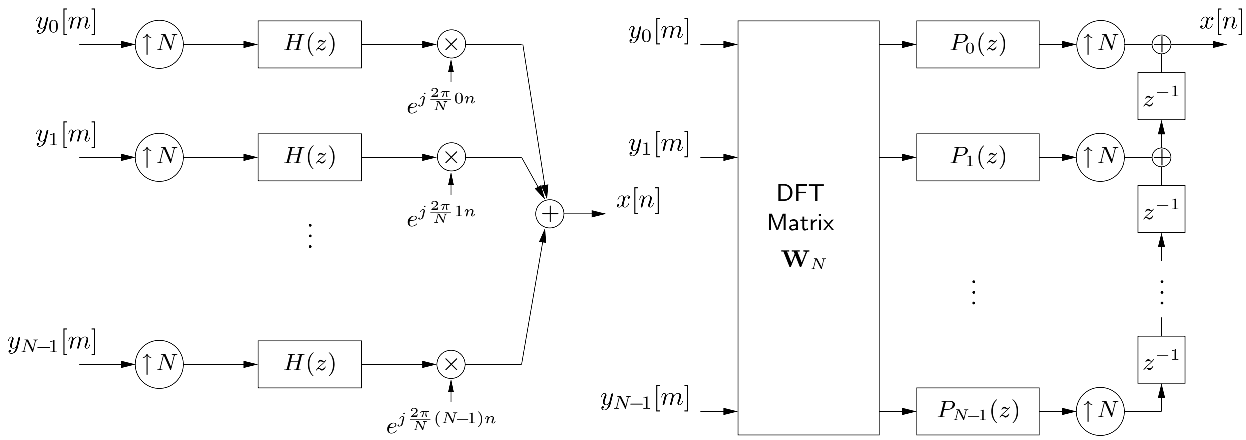
Recall that the previous block diagram represents the processing required by the k^{th} filterbank branch. Notice, however, that the polyphase filters $\{P_\ell(z)\}$ and the parallel-to-serial converter are common to all branches $k = 0, \ldots, N-1$; the difference between branches is determined only by the gains $e^{-j\frac{2\pi}{N}k\ell}$ on the left of the structure. Recall also that the system output is given by the branch sum

$$x[n] = \sum_{k=0}^{N-1} x_k[n].$$

Rather than summing $x_k[n]$ over k, we can sum $w_{\ell,k}[n]$ over k (for each ℓ):

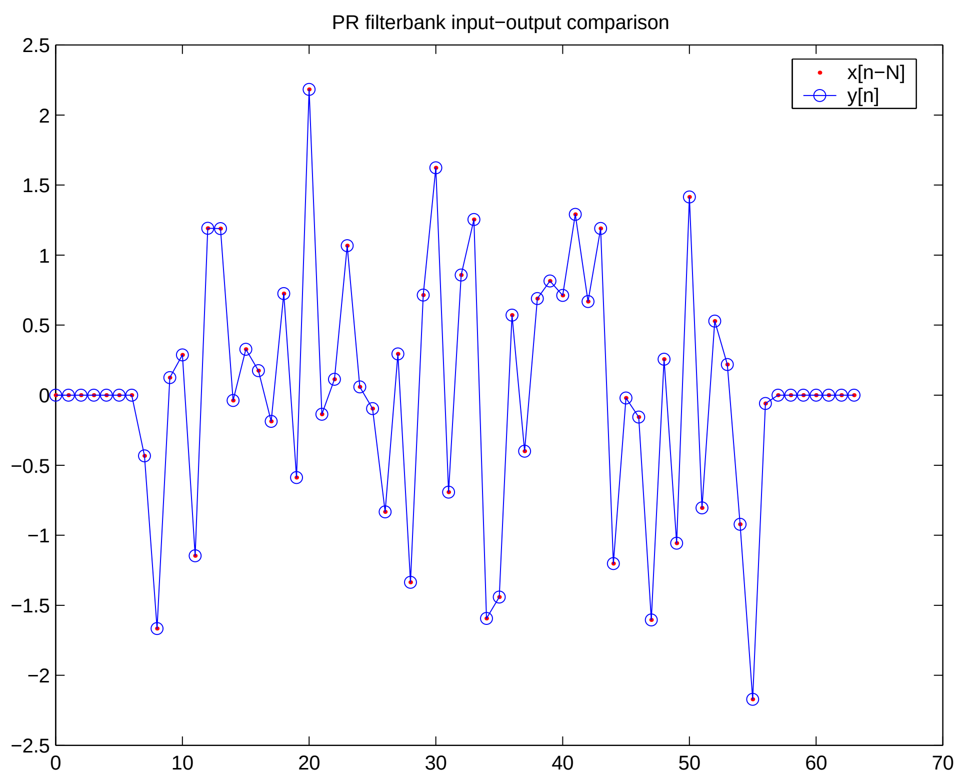
$$w_\ell[n] = \sum_{k=0}^{N-1} w_{\ell,k}[m] = \sum_{k=0}^{N-1} y_k[m] e^{-j\frac{2\pi}{N}\ell k}.$$

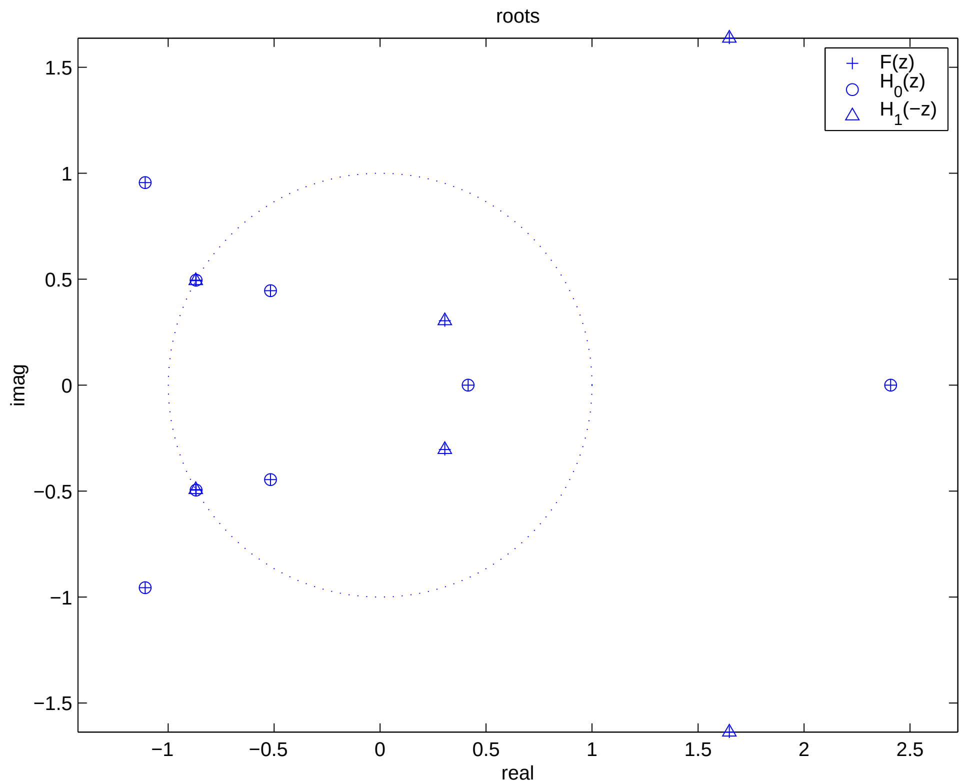
then apply the values $w_\ell[n]$ to a single polyphase reconstruction bank. Observe that the operation is equivalent to multiplication by a DFT matrix. The result looks like the diagram on the right below.

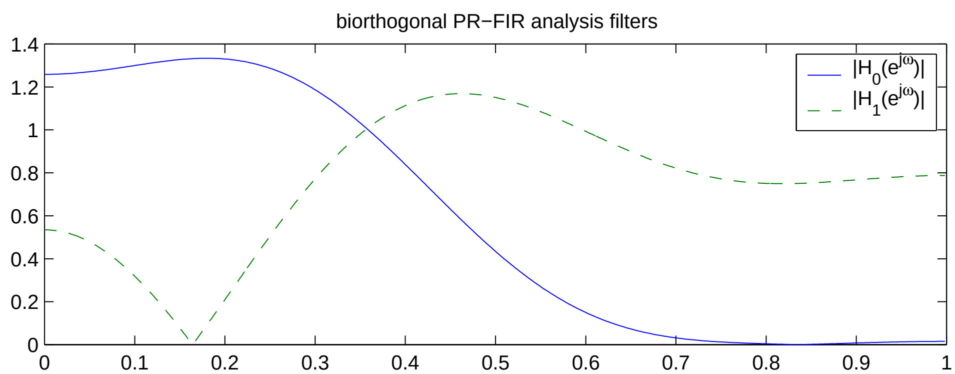


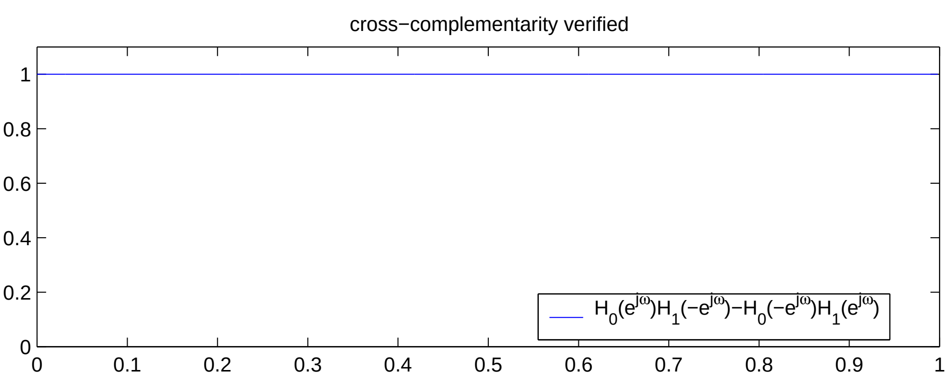
2. See plots on homework assignment.

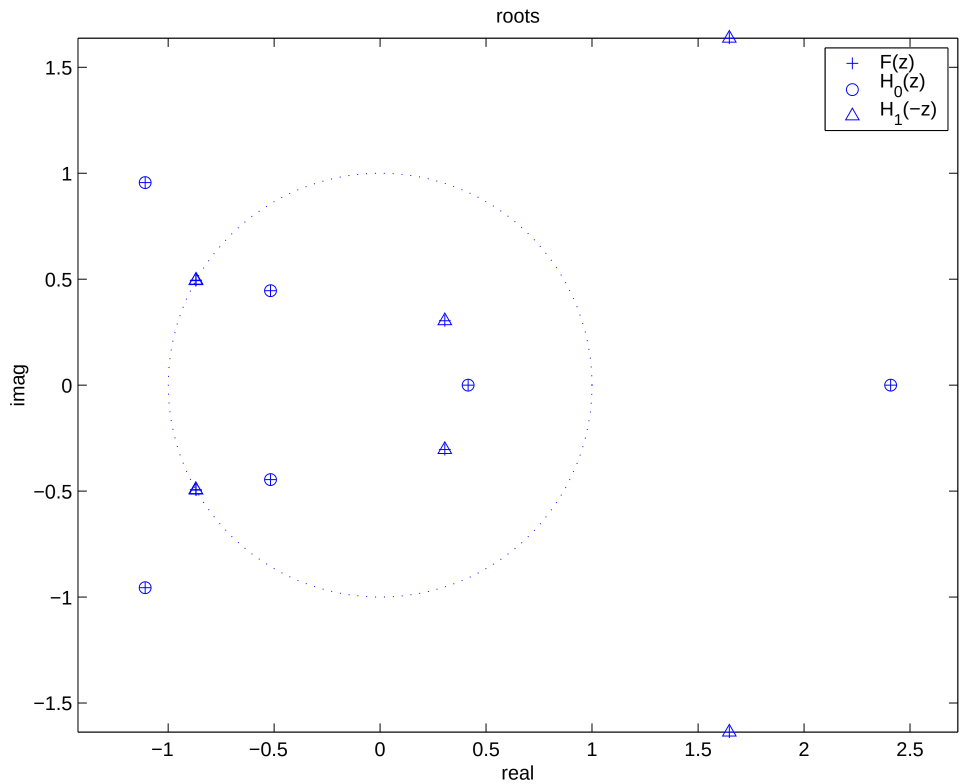
3. Bi-orthogonal FIR perfect reconstruction plots appear below. Plots are given for various root group allocations. In all cases, we get real-valued linear-phase perfectly reconstructing filters.

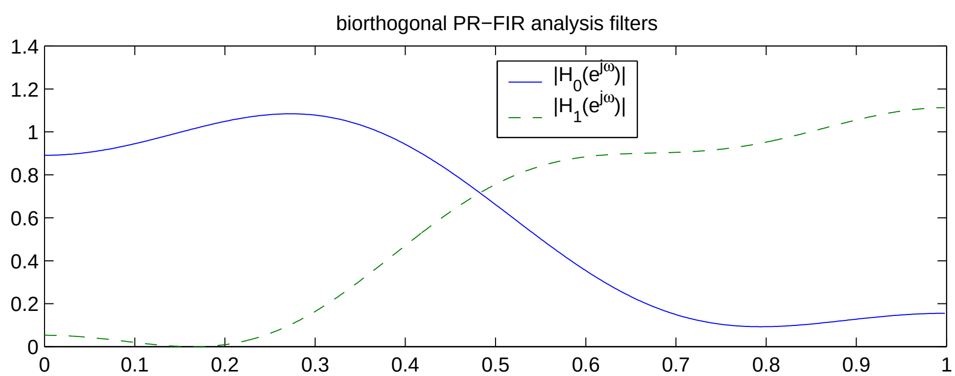


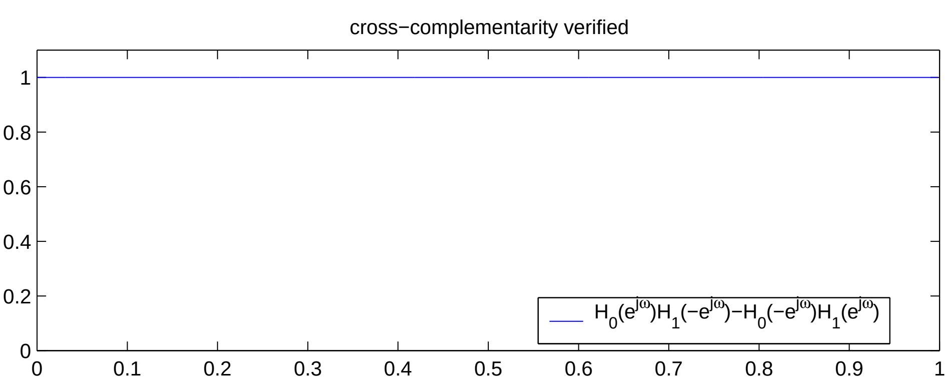

roots
F(z)
H0(z)
H1(−z)
imag
real

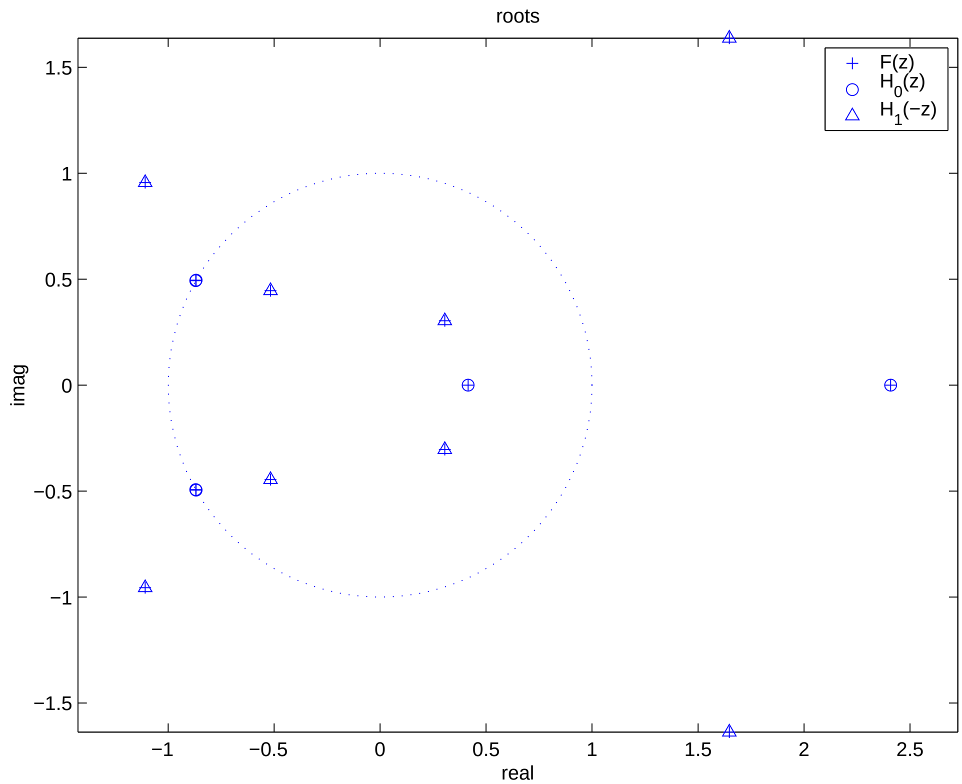

biorthogonal PR−FIR analysis filters
|H0(ejω)|
|H1(ejω)|

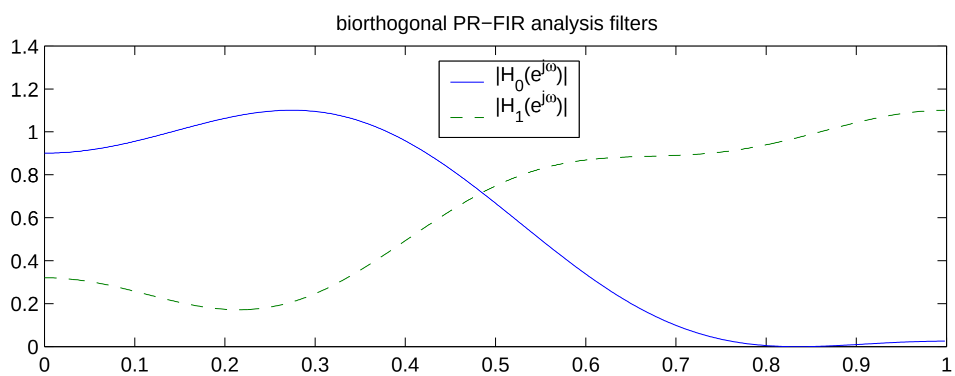

cross−complementarity verified
H0(ejω)H1(−ejω)−H0(−ejω)H1(ejω)

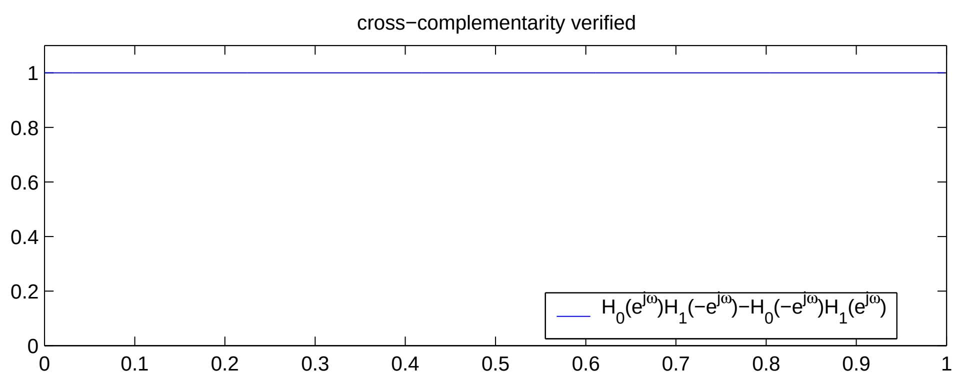

roots
F(z)
H0(z)
H1(−z)
imag
real
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5


biorthogonal PR−FIR analysis filters
|H0(ejω)|
|H1(ejω)|
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


cross−complementarity verified
H0(ejω)H1(−ejω)−H0(−ejω)H1(ejω)
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


roots
F(z)
$H_0(z)$
$H_1(-z)$
imag
real


biorthogonal PR−FIR analysis filters
$|H_0(e^{j\omega})|$
$|H_1(e^{j\omega})|$


cross−complementarity verified
$H_0(e^{j\omega})H_1(-e^{j\omega})-H_0(-e^{j\omega})H_1(e^{j\omega})$