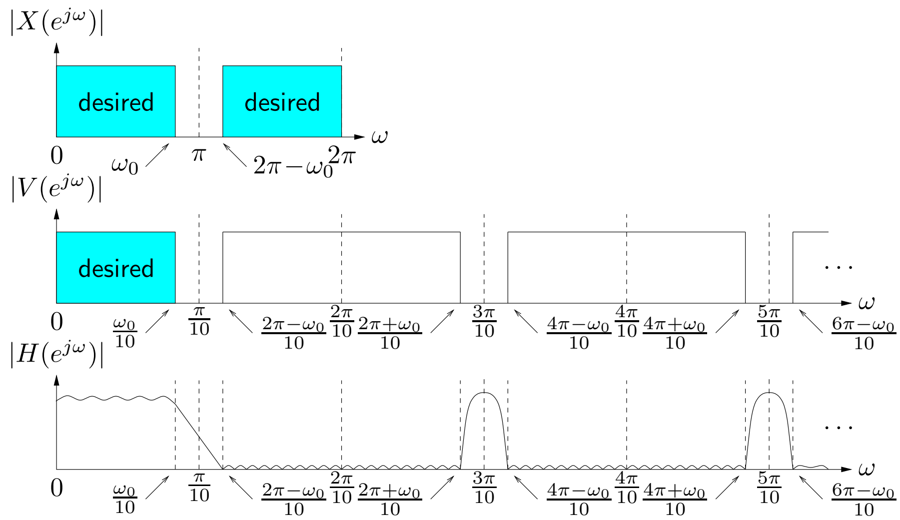
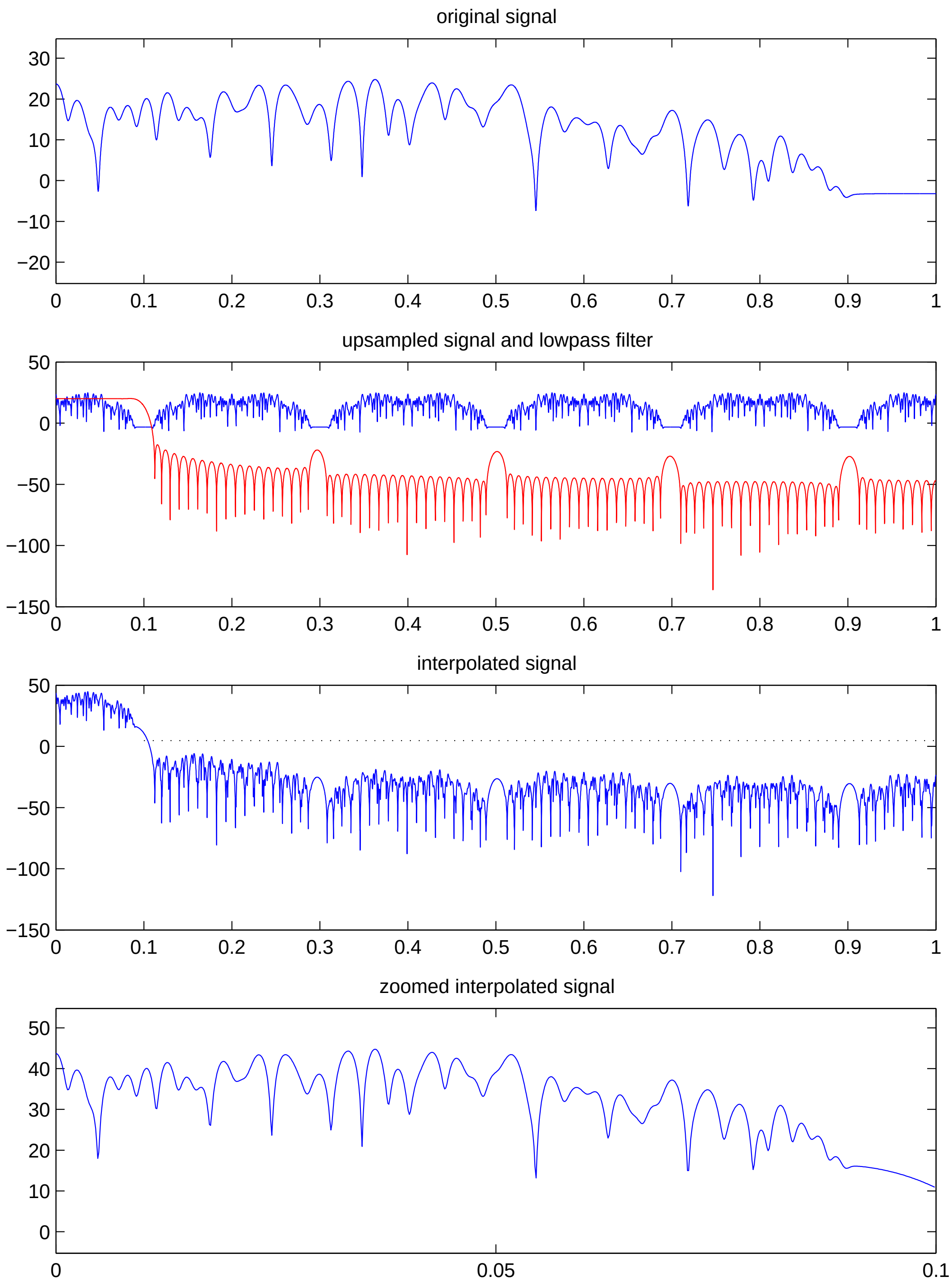
HOMEWORK SOLUTIONS #4

1. (b) In this problem, $L = 10$ and the input signal is bandlimited to $\omega = 0.9\pi$. For the Kaiser formula $\Delta\omega = \frac{2\pi-\omega_0}{L} - \frac{\omega_0}{L} = \frac{\pi}{50}$ and $\delta_1 = \delta_2 = 10^{-40/20} = 0.01$, giving $N_h = 185$. The filter design specs are shown below, where $v[m]$ denotes the upsampler output.

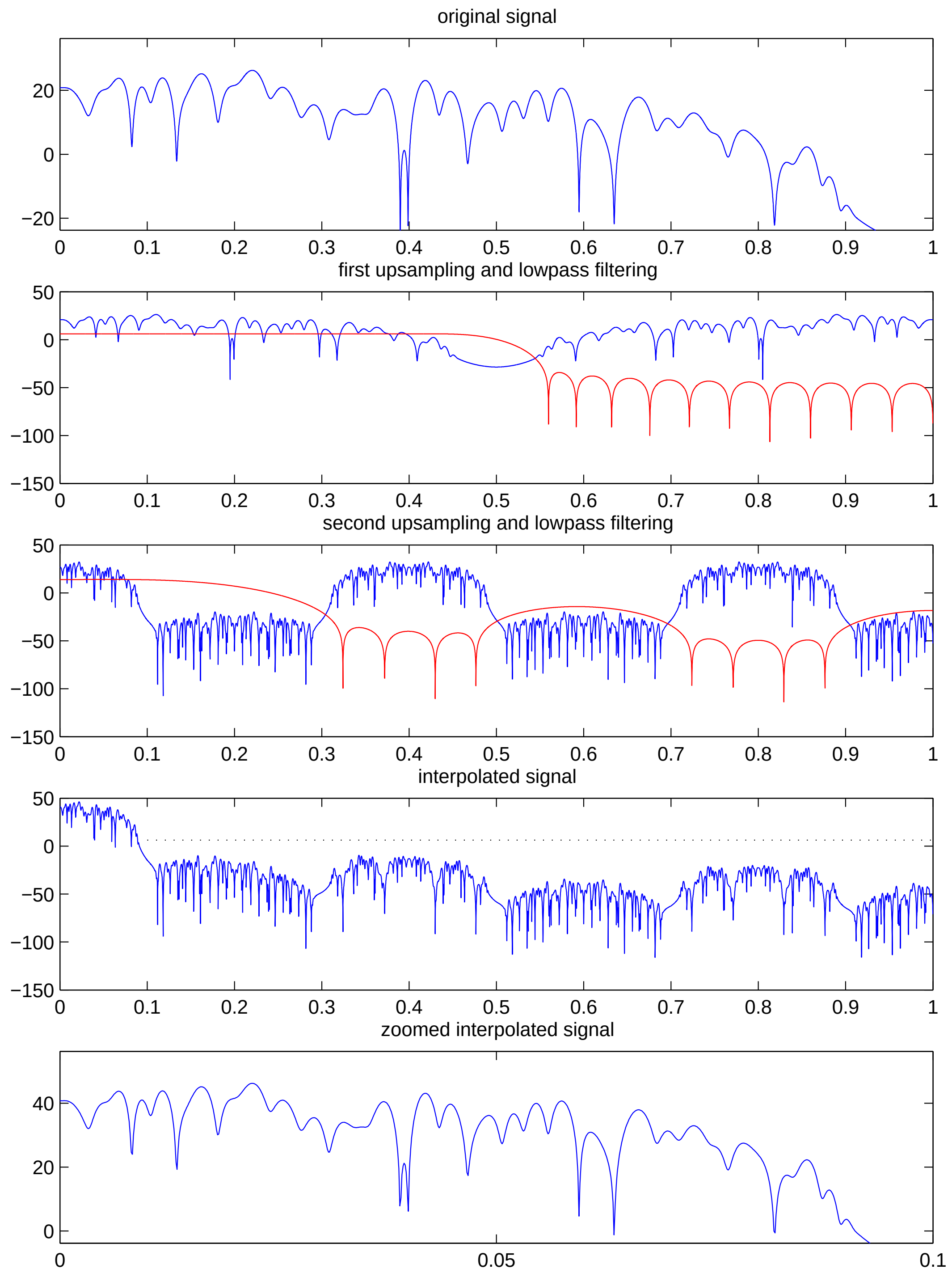


The plot on the following page shows the result of our Matlab simulation. The input signal, bandlimited to 0.9π, is shown on the top subplot. The next subplot shows the upsampled signal superimposed on the previously designed $H(e^{j\omega})$, where it is evident that the filter's transition bands are aligned with the spectral locations lacking energy. The third subplot shows the interpolated output, within which the dotted line shows that the unwanted images lie more than 40 dB below the desired spectral component. The final subplot zooms into the desired spectral component to show that it is a faithful representation of the input signal.

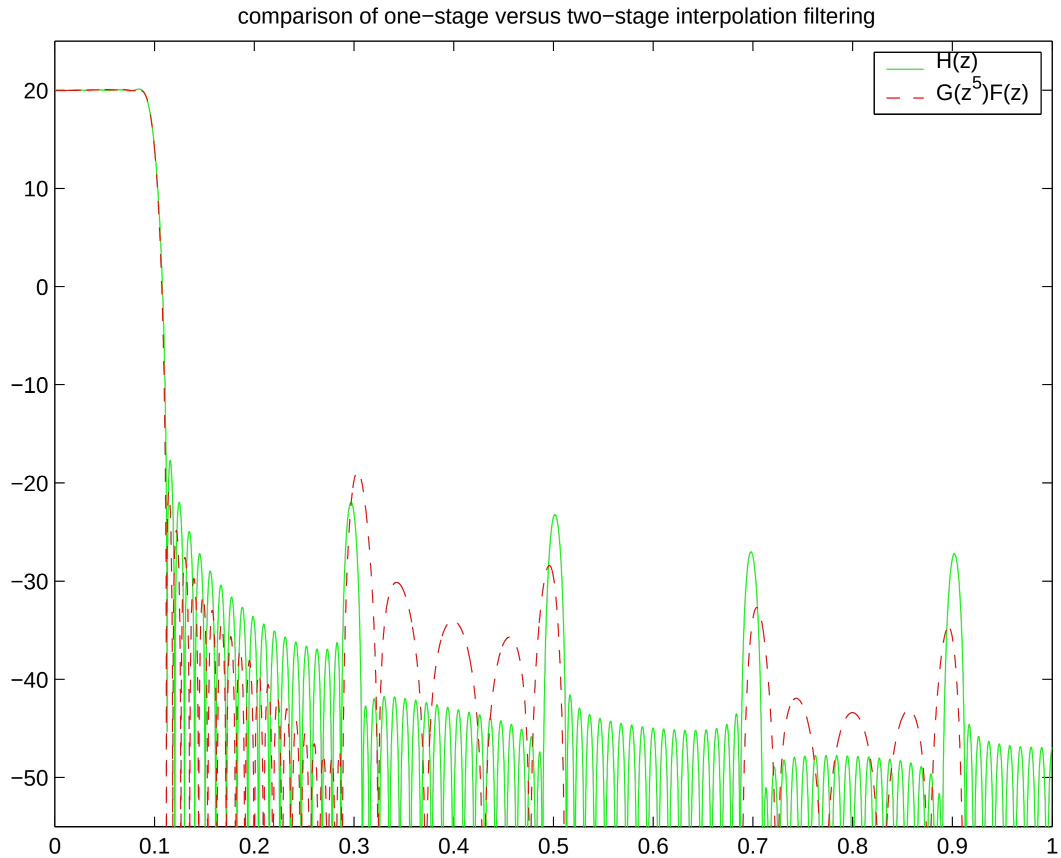

original signal
30
20
10
0
−10
−20
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
upsampled signal and lowpass filter
50
0
−50
−100
−150
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
interpolated signal
50
0
−50
−100
−150
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
zoomed interpolated signal
50
40
30
20
10
0
0 0.05 0.1

(c) The filter design procedure for two-stage interpolation is as follows: For $G(z)$, we have the passband edge at $\frac{\omega_0}{2}$ and the stopband edge at $\frac{2\pi-\omega_0}{2}$. leaving a single transition band centered at $\pi/2$. Thus, for calculation of N_g, we use $\Delta\omega = \frac{2\pi-\omega_0}{2} - \frac{\omega_0}{2} = \frac{\pi}{10}$, $\delta_1 = \frac{0.01}{2}$, and $\delta_2 = 0.01$, giving $N_g = 42$. For $F(z)$, we have the passband edge at $\frac{\omega_0/2}{5}$ and the first stopband edge at $\frac{2\pi-\omega_0/2}{5}$, which leaves the first transition band centered at $\frac{\pi}{5}$. Other transition bands exist centered at $\frac{3\pi}{5}$ and π. Thus, for calculation of N_f, we have $\Delta\omega = \frac{4\pi-\omega_0}{10} - \frac{\omega_0}{10} = \frac{11\pi}{50}$, $\delta_1 = \frac{0.01}{2}$, and $\delta_2 = 0.01$, giving $N_f = 19$.

The plot on the following page shows the result of our Matlab simulation. The input signal, bandlimited to 0.9π, is shown on the top subplot. The next subplot shows the upsampled-by-2 signal superimposed on $G(e^{j\omega})$, where it is evident that the filter's transition bands are aligned with the spectral locations lacking energy. The third subplot shows the upsampled-by-10 signal superimposed on $F(e^{j\omega})$, where again it is evident that the filter's transition bands are aligned with the spectral locations lacking energy. The fourth subplot shows the final output, within which the dotted line shows that the unwanted images lie more than 40 dB below the desired spectral component. The final subplot zooms into the desired spectral component to show that it is a faithful representation of the input signal.


original signal
20
0
−20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
first upsampling and lowpass filtering
50
0
−50
−100
−150
second upsampling and lowpass filtering
interpolated signal
zoomed interpolated signal
40
20
0
0.05
0.1

(d) The final plot shows a direct comparison of the effective LPFs for the two structures. The single-stage structure has LPF $H(z)$, while the two-stage structure has $G(z^5)F(z)$, where the impulse response of $G(z^5)$ is a 5-upsampled version of the impulse response of $G(z)$. We note that the effective LPFs are quite similar in character, but we know that the two-stage structure requires is more computationally efficient. Specifically, if we use polyphase implementations, the one-stage structure requires $N_h = 185$ MACs per input point, while the two-stage structure requires $N_g + 2N_f = 80$.



2. (a) When $L = 4$, $H_1(z^L)$ is a 4-upsampled version of the LPF $H_1(z)$ and has DTFT $H_1(e^{j\omega L})$. The top subplot below shows this DTFT.

The effectiveness of this structure hinges on the fact that $H_1(z^L)$ is a linear phase filter with group delay $\frac{L_1-1}{2}L$. This will ensure that $H_1(z^L)$ and $z^{-\frac{L_1-1}{2}L} - H_1(z^L)$ are complementary filters. To see why, recall that if $H_1(z)$ is linear phase with odd length L_1, then $H_1(z^L)$ is linear phase with odd length $L(L_1-1)+1$, meaning that its DTFT can be written

$$H_1(e^{j\omega L}) \;=\; \tilde{H}_1(e^{j\omega L})e^{-j\omega\frac{L_1-1}{2}L} \quad \text{for} \quad \tilde{H}_1(e^{j\omega L}) \in \mathbb{R}.$$

Using this result, $z^{-\frac{L_1-1}{2}L} - H_1(z^L)$ has the DTFT

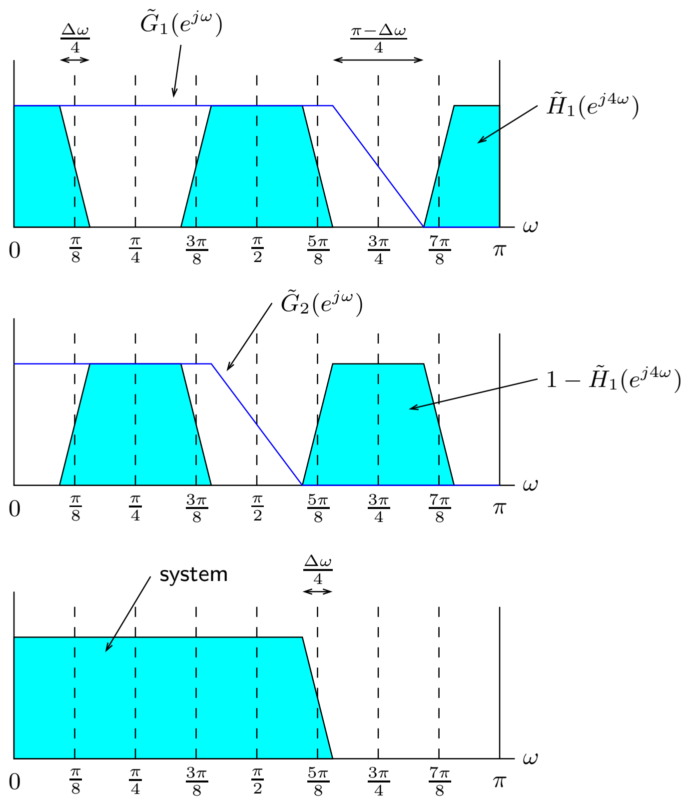
$$e^{-j\omega\frac{L_1-1}{2}L} - H_1(e^{j\omega L}) \;=\; \left(1 - \tilde{H}_1(e^{j\omega L})\right)e^{-j\omega\frac{L_1-1}{2}L}.$$

Note that passbands of $H_1(z^L)$ are stopbands of $z^{-\frac{L_1-1}{2}L} - H_1(z^L)$ and vice-a-versa. The middle subplot below shows the zero-phase component of this DTFT, i.e., $1 - \tilde{H}_1(e^{j\omega L})$.

Superimposing $G_1(e^{j\omega})$ and $G_2(e^{j\omega})$ onto the top and middle plots, respectively, we see that these filters are capable of removing the rightmost spectral images if their transition bands are less than $\frac{\pi-\Delta\omega}{4}$. The summed outputs of $G_1(z)$ and $G_2(z)$ will add coherently since $G_1(e^{j\omega})$ and $G_2(e^{j\omega})$ have the same phase response, a consequence of the fact that these filters are linear phase and of equal lengths. In other words,

$$\begin{aligned} G_1(e^{j\omega}) &= \tilde{G}_1(e^{j\omega})e^{-j\omega d} \quad \text{for some } d \text{ and} \quad \tilde{G}_1(e^{j\omega}) \in \mathbb{R} \\ G_2(e^{j\omega}) &= \tilde{G}_2(e^{j\omega})e^{-j\omega d} \quad \text{for some } d \text{ and} \quad \tilde{G}_2(e^{j\omega}) \in \mathbb{R} \end{aligned}$$

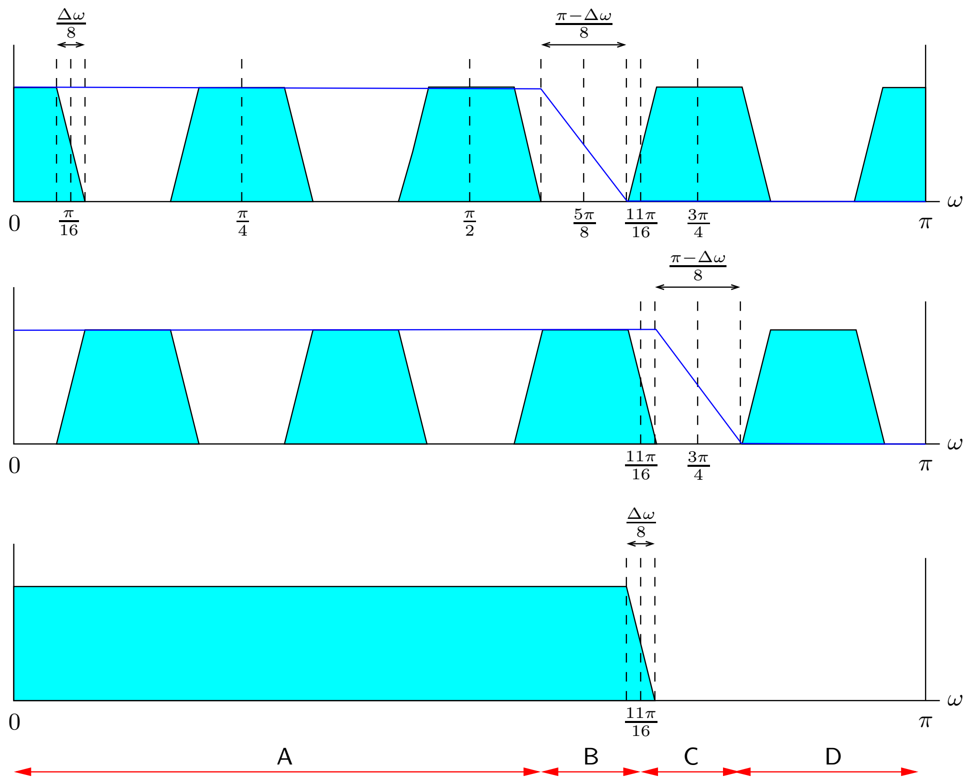
The result (i.e., the system output) is illustrated in the lower subplot. Note that the total system acts as a lowpass filter with cutoff at $\frac{5\pi}{8}$ and transition bandwidth $\frac{\Delta\omega}{4}$.



(b) For $L = 8$, images in $H_1(e^{jL\omega})$ repeat every multiple of $\frac{2\pi}{L} = \frac{\pi}{4}$ and have edge widths $\frac{\Delta\omega}{8}$, where $\Delta\omega$ is the transition bandwidth of $H_1(z)$. Since we would like the total system to have a transition bandwidth of 0.02π,

$$\frac{\Delta\omega}{8} = 0.02\pi \quad \Rightarrow \quad \Delta\omega = 0.16\pi$$

If we center the cutoff of $H_1(z)$ at $\frac{\pi}{2}$, the right edge of the first image in $H_1(e^{jL\omega})$ will be centered at $\frac{\pi}{16}$, and the left edge of the fourth image will be centered at $\frac{11\pi}{16}$. (See diagram below.) Then designing $G_1(z)$ to filter out the rightmost two images in its input spectrum and $G_2(z)$ to filter out the rightmost image in its input spectrum, we obtain the desired system response. For this we need the cutoff of $G_1(z)$ centered at $\frac{5\pi}{8}$ and the cutoff of $G_2(z)$ centered at $\frac{3\pi}{4}$, and we need that both $G_1(z)$ and $G_2(z)$ have transition bandwidths of $\frac{\pi-\Delta\omega}{8} = \frac{0.84\pi}{8}$.



To consider ripple performance, we need to consider four separate regions. We will work with the zero-phase components of $\tilde{H}_1(e^{jL\omega})$, $\tilde{G}_1(e^{j\omega})$, and $\tilde{G}_2(e^{j\omega})$.

A) In this region we are in the passband of G_1 and G_2, so we say that $\tilde{G}_1(e^{j\omega}) = 1 + \delta_p(\tilde{G}_1(e^{j\omega}))$ and $\tilde{G}_2(e^{j\omega}) = 1 + \delta_p(\tilde{G}_2(e^{j\omega}))$. Then the zero-phase component of the system output response is

$$\begin{aligned}
&\tilde{G}_1(e^{j\omega})\tilde{H}_1(e^{jL\omega}) + \tilde{G}_2(e^{j\omega})\big(1 - \tilde{H}_1(e^{jL\omega})\big) \\
&\quad = \big(1 + \delta_p(\tilde{G}_1(e^{j\omega}))\big)\tilde{H}_1(e^{jL\omega}) + \big(1 + \delta_p(\tilde{G}_2(e^{j\omega}))\big)\big(1 - \tilde{H}_1(e^{jL\omega})\big) \\
&\quad = 1 + \big(\delta_p(\tilde{G}_1(e^{j\omega})) - \delta_p(\tilde{G}_2(e^{j\omega}))\big)\tilde{H}_1(e^{jL\omega}) + \delta_p(\tilde{G}_2(e^{j\omega})) \\
&\quad = \begin{cases} 1 + \delta_p(\tilde{G}_2(e^{j\omega})) & \text{when } \tilde{H}_1(e^{jL\omega}) \approx 0 \\ 1 + \delta_p(\tilde{G}_1(e^{j\omega})) & \text{when } \tilde{H}_1(e^{jL\omega}) \approx 1 \end{cases}
\end{aligned}$$

Note that, in this region, the response (hence ripples) of H_1 cancel out!

B) In this region, we are in the transition band of $\tilde{G}_1$, the passband of $\tilde{G}_2$, and the stopband of $\tilde{H}_1$, so we say that $\tilde{G}_2(e^{j\omega}) = 1 + \delta_p(\tilde{G}_2(e^{j\omega}))$ and $\tilde{H}_1(e^{jL\omega}) = \delta_s(\tilde{H}_1(e^{jL\omega}))$. Then the zero-phase component of the system output response is

$$\begin{aligned}
&\tilde{G}_1(e^{j\omega})\tilde{H}_1(e^{jL\omega}) + \tilde{G}_2(e^{j\omega})\big(1 - \tilde{H}_1(e^{jL\omega})\big) \\
&\quad = \tilde{G}_1(e^{j\omega})\delta_s(\tilde{H}_1(e^{jL\omega})) + \big(1 + \delta_p(\tilde{G}_2(e^{j\omega}))\big)\big(1 - \delta_s(\tilde{H}_1(e^{jL\omega}))\big) \\
&\quad \approx \tilde{G}_1(e^{j\omega})\delta_s(\tilde{H}_1(e^{jL\omega})) + 1 + \delta_p(\tilde{G}_2(e^{j\omega})) - \delta_s(\tilde{H}_1(e^{jL\omega})) \\
&\quad = 1 + \big(\tilde{G}_1(e^{j\omega}) - 1\big)\delta_s(\tilde{H}_1(e^{jL\omega})) + \delta_p(\tilde{G}_2(e^{j\omega}))
\end{aligned}$$

Thus on the left side of this region, where $\tilde{G}_1(e^{j\omega}) \approx 1$, the stopband ripples in H_1 cancel out, leaving only the passband ripples of G_2. But on the right side of this region, where $\tilde{G}_1(e^{j\omega}) \approx 0$, the stopband ripples of H_1 are not attenuated, and they may add to the passband ripples of G_1. This is confirmed in the matlab plots, where it is easily seen that the ripples get worse towards the right edge of this region.

C) In this region, we are in the stopband of $\tilde{G}_1$, the transition band of $\tilde{G}_2$, and the passband of H_1, so we say that $\tilde{G}_1(e^{j\omega}) = \delta_s(\tilde{G}_1(e^{j\omega}))$ and $\tilde{H}_1(e^{jL\omega}) = 1 + \delta_p(\tilde{H}_1(e^{jL\omega}))$. Then the zero-phase component of the system output response is

$$\begin{aligned}
&\tilde{G}_1(e^{j\omega})\tilde{H}_1(e^{jL\omega}) + \tilde{G}_2(e^{j\omega})\big(1 - \tilde{H}_1(e^{jL\omega})\big) \\
&\quad = \delta_s(\tilde{G}_1(e^{j\omega}))\big(1 + \delta_p(\tilde{H}_1(e^{jL\omega}))\big) - \tilde{G}_2(e^{j\omega})\delta_p(\tilde{H}_1(e^{jL\omega})) \\
&\quad \approx \delta_s(\tilde{G}_1(e^{j\omega})) - \tilde{G}_2(e^{j\omega})\delta_p(\tilde{H}_1(e^{jL\omega}))
\end{aligned}$$

Thus on the left side of this region, where $\tilde{G}_2(e^{j\omega}) \approx 1$, the passband ripples in H_1 are not attenuated, and they may add to the stopband ripples of G_1. But on the right side of this region, where $\tilde{G}_2(e^{j\omega}) \approx 0$, the passband ripples of H_1 are attenuated, leaving only the stopband ripples of G_1. This is confirmed in the matlab plots, where it is easily seen that the ripples get worse towards the right edge of this region.

D) In this region we are in the stopband of G_1 and G_2, so we say that $\tilde{G}_1(e^{j\omega}) = \delta_s(\tilde{G}_1(e^{j\omega}))$ and $\tilde{G}_2(e^{j\omega}) = \delta_s(\tilde{G}_2(e^{j\omega}))$. Then the zero-phase component of the system output response is

$$\begin{aligned}
&\tilde{G}_1(e^{j\omega})\tilde{H}_1(e^{jL\omega}) + \tilde{G}_2(e^{j\omega})\big(1 - \tilde{H}_1(e^{jL\omega})\big) \\
&\quad = \delta_s(\tilde{G}_1(e^{j\omega}))\tilde{H}_1(e^{jL\omega}) + \delta_s(\tilde{G}_2(e^{j\omega}))\big(1 - \tilde{H}_1(e^{jL\omega})\big) \\
&\quad = \begin{cases} \delta_s(\tilde{G}_2(e^{j\omega})) & \text{when } \tilde{H}_1(e^{jL\omega}) \approx 0 \\ \delta_s(\tilde{G}_1(e^{j\omega})) & \text{when } \tilde{H}_1(e^{jL\omega}) \approx 1 \end{cases}
\end{aligned}$$

Note that, in this region also, the response (hence ripples) of H_1 cancel out!

Let's now summarize our findings. Throughout most of the total system's passband, the ripple is determined alternately by the passband ripples of G_1 and G_2. Thus we should set them equal, i.e., $\delta_p(\tilde{G}_1(e^{j\omega})) = \delta_p(\tilde{G}_2(e^{j\omega}))$. Likewise, throughout most of the total system's stopband, the ripple is determined alternately by the stopband ripples of G_1 and G_2. Thus we should set them equal, i.e., $\delta_s(\tilde{G}_1(e^{j\omega})) = \delta_s(\tilde{G}_2(e^{j\omega}))$. Next, the edge of the total system's passband has the ripple contributions from both $\delta_s(\tilde{H}_1(e^{jL\omega}))$ and $\delta_p(\tilde{G}(e^{j\omega}))$. Similarly, the edge of the total system's stopband has the ripple contributions from both $\delta_p(\tilde{H}_1(e^{jL\omega}))$ and $\delta_s(\tilde{G}(e^{j\omega}))$. If we desire that the total system has both passband ripple and stopband ripple equal to $\delta = 0.05$, we could set

$$\delta_s(\tilde{H}_1(e^{jL\omega})) = \delta_p(\tilde{H}_1(e^{jL\omega})) = \delta_s(\tilde{G}(e^{j\omega})) = \delta_p(\tilde{G}(e^{j\omega})) = \frac{\delta}{2} = 0.025.$$

(c) Straightforward FIR lowpass:

$$N \approx \frac{-10\log_{10}(\delta_s\delta_p)-13}{2.3\cdot\Delta\omega} = \frac{-10\log_{10}(0.05^2)-13}{2.3\cdot 0.02\pi} \approx 90$$

New structure:

$$N_{G_1} = N_{G_2} \approx \frac{-10\log_{10}(0.025^2)-13}{2.3\cdot\frac{0.84\pi}{8}} \approx 25$$

$$N_{H_1} \approx \frac{-10\log_{10}(0.025^2)-13}{2.3\cdot 0.16\pi} \approx 17$$

So the total number of multiplications per output for the new structure is $\approx 17 + 2\cdot 25 = 67$. Actually, H_1 is a "halfband" filter and so about half of its coefficients are zero, making the real total ≈ 60.

(d) See the plot below.