HOMEWORK SOLUTIONS #2

1. (a) Denoting the signal after the downsampler by $v[m]$, we see that

$$\begin{aligned} V(z) &= \frac{1}{M}\sum_{p=0}^{M-1} X(e^{-j\frac{2\pi}{M}p} z^{\frac{1}{M}}) \\ Y(z) &= V(z^M) \\ &= \frac{1}{M}\sum_{p=0}^{M-1} X(e^{-j\frac{2\pi}{M}p} z) \\ Y(e^{j\omega}) &= \frac{1}{M}\sum_{p=0}^{M-1} X\left(e^{j\left(\omega-\frac{2\pi}{M}p\right)}\right) \end{aligned}$$

Note that we made the substitution $z = e^{j\omega}$ for the last step.

(b) Denoting the signal after the delay by $v[m]$, we see that

$$\begin{aligned} V(z) &= z^{-2}\frac{1}{M}\sum_{p=0}^{M-1} X(e^{-j\frac{2\pi}{M}p} z^{\frac{1}{M}}) \\ Y(z) &= V(z^M) \\ &= z^{-2M}\frac{1}{M}\sum_{p=0}^{M-1} X(e^{-j\frac{2\pi}{M}p} z) \\ Y(e^{j\omega}) &= e^{-j2M\omega}\frac{1}{M}\sum_{p=0}^{M-1} X\left(e^{j\left(\omega-\frac{2\pi}{M}p\right)}\right) \end{aligned}$$

(c) Here zeros are inserted between the samples of $x[n]$, but then the same zeros are discarded, so that $x[n] = y[n]$ and $Y(e^{j\omega}) = X(e^{j\omega})$. This can be verified in the z domain as follows. Denoting the signal after the upsampler by $v[m]$, we see that

$$\begin{aligned} V(z) &= X(z^M) \\ Y(z) &= \frac{1}{M}\sum_{p=0}^{M-1} V(e^{-j\frac{2\pi}{M}p} z^{\frac{1}{M}}) \\ &= \frac{1}{M}\sum_{p=0}^{M-1} X(e^{-j2\pi p} z) \\ &= X(z) \end{aligned}$$

where we use the fact that $e^{-j2\pi p} = 1$ for all integer p.

(d) Since $M \geq 3$, the upsampled sequence is delayed such that the downsampler retains only zeros. Thus, $Y(e^{j\omega}) = 0$. This can be verified in the z domain as follows. Denoting the signal after the delay by $v[m]$, we see that

$$\begin{aligned} V(z) &= z^{-2}X(z^M) \\ Y(z) &= \frac{1}{M}\sum_{p=0}^{M-1} V(e^{-j\frac{2\pi}{M}p}z^{\frac{1}{M}}) \\ &= \frac{1}{M}\sum_{p=0}^{M-1} (e^{-j\frac{2\pi}{M}p}z^{\frac{1}{M}})^{-2}X(e^{-j2\pi p}z) \\ &= X(z)z^{-\frac{2}{M}} \underbrace{\frac{1}{M}\sum_{p=0}^{M-1} e^{j\frac{2\pi}{M}2p}}_{=0} \end{aligned}$$

where we have used the fact that $M > 2$ in the last step.

2. The black box could be constructed to interleave the three sampled sequences as in the figure below

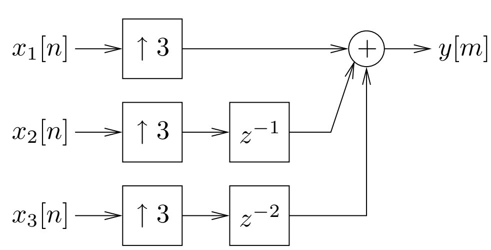
$$\ldots, x_1[0], x_2[0], x_3[0], x_1[1], x_2[1], x_3[1], x_1[2], x_2[2], x_3[2], \ldots$$

which is equal to

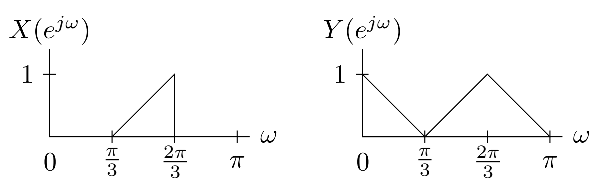
$$\ldots, x_c(0), x_c(T/3), x_c(2T/3), x_c(T), x_c(4T/3), x_c(5T/3), x_c(2T), x_c(7T/3), x_c(8T/3), \ldots$$

so that $y[m] = x_c(mT/3)$ for $m \in \mathbb{Z}$. Thus $y[m]$ is a uniformly-sampled version of $x_c(t)$ with sampling rate $3/T$. Since $x_c(t)$ is bandlimited to $1.25/T$, any sampling rate higher than $2.5/T$ is sufficient to prevent aliasing. Thus, $x_c(t)$ can be perfectly reconstructed using the standard sinc-interpolation

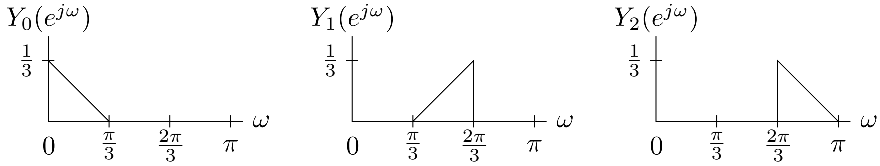
$$x_c(t) = \sum_{m=-\infty}^{\infty} y[m] \frac{\sin\left(\frac{3\pi}{T}(t - m\frac{T}{3})\right)}{\frac{3\pi}{T}(t - m\frac{T}{3})}$$



3. Since all sequences and coefficients are real-valued, all DTFTs are conjugate symmetric. Using this fact and the results of problem 1, $Y(e^{j\omega})$ has the form below.



The filters $H_k(e^{j\omega})$ isolate various parts of the spectrum, as below.



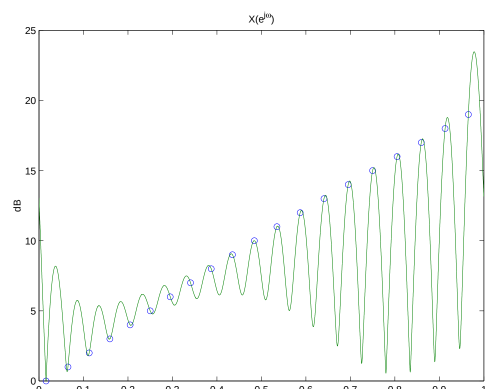
4. Since

$$X(e^{j\omega_k}) \;=\; \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_k n}$$

we can construct a generalized DFT matrix using $W_k = e^{-j\omega_k}$:

$$\underbrace{\begin{pmatrix} X(e^{j\omega_0}) \\ X(e^{j\omega_1}) \\ \vdots \\ X(e^{j\omega_{N-1}}) \end{pmatrix}}_{\mathbf{X}} \;=\; \underbrace{\begin{pmatrix} W_0^0 & W_0^1 & W_0^2 & W_0^3 & \dots \\ W_1^0 & W_1^1 & W_1^2 & W_1^3 & \dots \\ W_2^0 & W_2^1 & W_2^2 & W_2^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{\mathbf{W}} \underbrace{\begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix}}_{\mathbf{x}}$$

Due to the Vandermonde structure of this matrix, we know that if $\omega_k \neq \omega_\ell$ for $k \neq \ell$, the matrix is invertible. Thus $\mathbf{x} = \mathbf{W}^{-1}\mathbf{X}$. Using $X(e^{j\omega_k}) = k$ and $\omega_k = 2\pi k/N + 0.1\cos(2\pi k/N)$ for $k = 0, 1, \dots, N-1$, we generate the following plot:



5. The following code implemented our interpolator, whose plots where given in the problem statement.

```
% solution for ece700 homework 2 problem 5

 % generate bandlimited signal
 N_x = 1000;
 N_g = 51;
 u = randn(1,N_x+N_g);
 g = firpm(N_g-1,[0,.6,0.8,1],[1,1,0,0]);
 x = conv(u,g);
 x = x(1+(N_g-1)/2:N_x+(N_g-1)/2);

 % design interpolation filter
 N_h = 33;
 h = firpm(N_h-1,[0,8/30,12/30,28/30],[3,3,0,0]);

 % upsample and filter
 v = zeros(1,3*length(x)); v(1:3:end) = x;
 y = conv(v,h);

 % plot spectra
 M = 2^ceil(log2(length(y))+2); % dft length
 figure(1)
 subplot(221)
   plot(2*[0:M-1]/M,abs(fft(u,M))); title('fullband signal u[n]');
 subplot(222)
   plot(2*[0:M-1]/M,abs(fft(x,M))); title('filtered signal x[n]');
 subplot(223)
   plot(2*[0:M-1]/M,abs(fft(v,M))); title('upsampled signal v[m]');
 subplot(224)
   plot(2*[0:M-1]/M,abs(fft(y,M))); title('interpolated signal y[m]');

 % plot time domain
 figure(2)
 n1_x = 500; n2_x = 510;
 m1_y = 3*(n1_x-1)+1 + (N_h-1)/2; m2_y = 3*(n2_x-1)+1 + (N_h-1)/2;
 plot([n1_x:1/3:n2_x],y([m1_y:m2_y]),'ro');
 hold on; stem([n1_x:n2_x],x([n1_x:n2_x]),'x'); hold off;
 title('standard interpolation: original (x) and interpolated (o) values');
```