HOMEWORK SOLUTIONS #1

1. Since periodic with period T, can represent $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ using a Fourier series with coefficients $P[k]$:

$$p(t) \;=\; \sum_{k=-\infty}^{\infty} P[k] e^{jk\frac{2\pi}{T}t}$$

where

$$\begin{aligned} P[k] &:= \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jk\frac{2\pi}{T}t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t-nT) e^{-jk\frac{2\pi}{T}t} dt \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} \delta(t-nT) e^{-jk\frac{2\pi}{T}t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\frac{2\pi}{T}t} dt \\ &= \frac{1}{T} \end{aligned}$$

due to the sifting property of the Dirac delta. Thus we have

$$p(t) \;=\; \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

2. First note that

$$y_c(t) \;=\; \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(2000t - n))}{\pi(2000t-n)} \;=\; \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\frac{\pi}{T}(t-nT))}{\frac{\pi}{T}(t-nT)}\bigg|_{\frac{1}{T}=2000}$$

where $\frac{\sin(\frac{\pi}{T}t)}{\frac{\pi}{T}t}$ is the impulse response of an ideal reconstruction filter for a rate-$\frac{1}{T}$ sampled signal. Thus, if $x_c(t)$ was bandlimited to $\frac{1}{2T}$ =1 kHz, we would have $y_c(t) = x_c(t)$. So all components of $x_c(t)$ bandlimited to 1 kHz will appear un-aliased in $y_c(t)$, while all higher-frequency components of $x_c(t)$ will be aliased to the range between -1 kHz and 1 kHz (due to the linearity of the DTFT).

   (a) The cosines at 300 Hz and 500 Hz pass appear un-aliased at the output, whereas the higher frequencies are aliased:

$$\begin{aligned} \cos(2\pi \cdot 1200 \cdot \tfrac{n}{2000}) &= \cos(2\pi \cdot (-800) \cdot \tfrac{n}{2000}) \\ \cos(2\pi \cdot 1700 \cdot \tfrac{n}{2000}) &= \cos(2\pi \cdot (-300) \cdot \tfrac{n}{2000}) \\ \cos(2\pi \cdot 5500 \cdot \tfrac{n}{2000}) &= \cos(2\pi \cdot (-500) \cdot \tfrac{n}{2000}) \end{aligned}$$

   Then, since $\cos(t) = \cos(-t)$,

$$y_c(t) \;=\; 2\cos(2\pi \cdot 300 \cdot t) + 2\cos(2\pi \cdot 500 \cdot t) + \cos(2\pi \cdot 800 \cdot t)$$

(b) Repeating above with cosine replaced by sine,

$$\begin{aligned}\sin(2\pi \cdot 1200 \cdot \tfrac{n}{2000}) &= \sin(2\pi \cdot (-800) \cdot \tfrac{n}{2000})\\ \sin(2\pi \cdot 1700 \cdot \tfrac{n}{2000}) &= \sin(2\pi \cdot (-300) \cdot \tfrac{n}{2000})\\ \sin(2\pi \cdot 5500 \cdot \tfrac{n}{2000}) &= \sin(2\pi \cdot (-500) \cdot \tfrac{n}{2000})\end{aligned}$$

Then, using $\sin(t) = -\sin(-t)$, we find that the signal components at ± 300 and ± 500 Hz cancel, giving

$$y_c(t) = -\sin(2\pi \cdot 800 \cdot t)$$

(c) Repeating with complex exponentials,

$$\begin{aligned}e^{j2\pi \cdot 1200 \cdot \frac{n}{2000}} &= e^{j2\pi \cdot (-800) \cdot \frac{n}{2000}}\\ e^{j2\pi \cdot 1700 \cdot \frac{n}{2000}} &= e^{j2\pi \cdot (-300) \cdot \frac{n}{2000}}\\ e^{j2\pi \cdot 5500 \cdot \frac{n}{2000}} &= e^{j2\pi \cdot (-500) \cdot \frac{n}{2000}}\end{aligned}$$

so that

$$\begin{aligned}y_c(t) &= e^{j2\pi \cdot 300 \cdot t} + e^{j2\pi \cdot (-300) \cdot t} + e^{j2\pi \cdot 500 \cdot t} + e^{j2\pi \cdot (-500) \cdot t} + e^{j2\pi \cdot (-800) \cdot t}\\ &= 2\cos(2\pi \cdot 300 \cdot t) + 2\cos(2\pi \cdot 500 \cdot t) + e^{-j2\pi \cdot 800 \cdot t}\end{aligned}$$

3. (a) From the sampling theorem, $y_1[n] = y_c(nT)$ implies

$$Y_1(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega - 2\pi k}{T}\Big)\right)$$

Since we require band-limiting to $\frac{1}{2T}$ to guarantee no aliasing, there may be aliasing in $Y_1(e^{j\omega})$.

(b) If we define $\tilde{y}_c(t) = y_c(t + T/2)$, then $y_2[n] = y_c(nT + T/2) = \tilde{y}_c(nT)$, and the sampling theorem implies

$$Y_2(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \tilde{Y}_c\left(j\Big(\frac{\omega - 2\pi k}{T}\Big)\right)$$

where $\tilde{Y}_c$ denotes the CTFT of $\tilde{y}_c(t)$. Since y_c and $\tilde{y}_c$ are related by a time shift of $T/2$, their CTFTs are related as follows:

$$\begin{aligned}\tilde{Y}_c(j\Omega) &= \int_{-\infty}^{\infty} y_c(t + T/2) e^{-j\Omega t} dt\\ &= \int_{-\infty}^{\infty} y_c(\tau) e^{-j\Omega(\tau - T/2)} d\tau\\ &= e^{j\Omega T/2} \int_{-\infty}^{\infty} y_c(\tau) e^{-j\Omega \tau} d\tau\\ &= e^{j\Omega T/2} Y_c(j\Omega)\end{aligned}$$

Thus

$$\begin{aligned}Y_2(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega - 2\pi k}{T}\Big)\right) e^{j(\frac{\omega - 2\pi k}{T})\frac{T}{2}}\\ &= e^{j\frac{\omega}{2}} \frac{1}{T} \sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega - 2\pi k}{T}\Big)\right) (-1)^k\end{aligned}$$

$Y_2(e^{j\omega})$ may contain aliasing for the same reasons as in part (a).

(c) Combining the results of the previous two parts,

$$\begin{aligned} Y_3(e^{j\omega/2}) &= Y_1(e^{j\omega}) + e^{-j\frac{\omega}{2}} Y_2(e^{j\omega}) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega-2\pi k}{T}\Big)\right) + \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega-2\pi k}{T}\Big)\right)(-1)^k \\ &= \frac{2}{T}\sum_{k \text{ even}} Y_c\left(j\Big(\frac{\omega-2\pi k}{T}\Big)\right) \end{aligned}$$

since the odd-indexed elements cancel. Continuing,

$$\begin{aligned} Y_3(e^{j\omega/2}) &= \frac{2}{T}\sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega-4\pi k}{T}\Big)\right) \\ &= \frac{1}{T/2}\sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega/2-2\pi k}{T/2}\Big)\right) \end{aligned}$$

(d) Replacing $\omega/2$ with ω we get

$$Y_3(e^{j\omega}) = \frac{1}{T/2}\sum_{k=-\infty}^{\infty} Y_c\left(j\Big(\frac{\omega-2\pi k}{T/2}\Big)\right)$$

Note that $Y_3(e^{j\omega})$ is the DTFT that would have been obtained if we would have sampled $y_c(t)$ at rate $2/T$. In other words, if $y_3[m]$ is the inverse-DTFT of $Y_3(e^{j\omega})$, then $y_3[m] = y_c(mT/2)$. Since, for $2/T$-rate sampling, a signal bandlimited to $1/T$ Hz will not alias, we conclude that there is no aliasing present in $y_3[m]$.

As will become clear when we are more familiar with multirate DSP, $y_3[m]$ results when the sequences $y_1[n]$ and $y_2[n]$ are interleaved.

4. (a) Define $\tilde{x}_c(t) = x_c(t - \tau T)$. Then say $\tilde{x}[n] = \tilde{x}_c(nT) = x_c(nT - \tau T)$. We want to show, in the end, that $y[n] = \tilde{x}[n]$. We know from the sampling theorem that

$$\tilde{X}(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \tilde{X}_c\left(j\Big(\frac{\omega-2\pi k}{T}\Big)\right)$$

We also know from the CTFT time shift property (or as shown in the solution to 3(b)) that

$$\tilde{X}_c(j\Omega) = e^{-j\Omega\tau T} X_c(j\Omega)$$

so that

$$\begin{aligned} \tilde{X}(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\Big(\frac{\omega-2\pi k}{T}\Big)\right) e^{-j(\frac{\omega-2\pi k}{T})\tau T} \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\Big(\frac{\omega-2\pi k}{T}\Big)\right) e^{-j(\omega-2\pi k)\tau} \end{aligned}$$

As a DTFT, recall that $\tilde{X}(e^{j\omega})$ is completely specified by its values in the range $\omega \in [-\pi, \pi)$, so let's examine $\tilde{X}(e^{j\omega})$ over this range. Inferring from the problem statement that $x_c(t)$ must have been bandlimited to $\frac{\pi}{T}$ rad/s, we can claim

$$\tilde{X}(e^{j\omega}) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right) e^{-j\omega\tau}, \quad |\omega| < \pi$$

Band-limiting also implies that

$$X(e^{j\omega}) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

so that

$$\tilde{X}(e^{j\omega}) = X(e^{j\omega})e^{-j\omega\tau}, \quad |\omega| < \pi$$

Now consider $y[n]$. From the convolution property of the DTFT,

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\ &= X(e^{j\omega})e^{-j\omega\tau}, \quad |\omega| < \pi \end{aligned}$$

where the second equation follows from the problem statement. Since $Y(e^{j\omega}) = \tilde{X}(e^{j\omega})$ for all $|\omega| < \pi$, the same must hold true for all ω, from which it follows that $y[n] = \tilde{x}[n] = x\big((n-\tau)T\big)$. Basically, the filter $H(z)$ imposes a *possibly non-integer* delay of τ samples!

(b) From the inverse DTFT,

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\tau)} d\omega \\ &= \frac{1}{2\pi}\frac{1}{j(n-\tau)}\left[e^{j\pi(n-\tau)} - e^{-j\pi(n-\tau)}\right] \\ &= \frac{1}{\pi(n-\tau)}\sin(\pi(n-\tau)) \\ &= \mathrm{sinc}(\pi(n-\tau)) \end{aligned}$$

Note that $h[n]$ has a doubly infinite number of coefficients. Therefore, it is impossible to implement exactly. However, we can approximate it using short FIR filters, as shown below.

(c) We see by the plot below that the filter closely approximates the desired group delay over the frequency range $\omega \in [0, 0.6\frac{\pi}{2T})$. Thus, we expect the filter to perform well on lowpass signals.

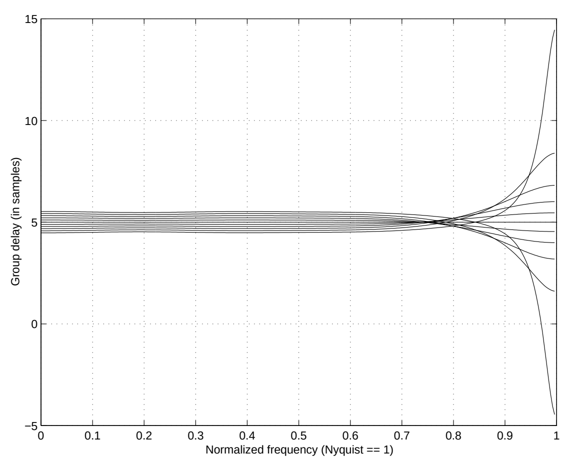

Figure 1: Group delay of length-11 windowed approximation to $h[n]$.