ECE-700 – Review

Phil Schniter

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1 Transforms

Using $x_c(t)$ to denote a continuous-time signal at time $t \in \mathbb{R}$,

- *Laplace Transform:*

$$X_c(s) = \int_{-\infty}^{\infty} x_c(t)e^{-st}dt, \quad s \in \mathbb{C}$$

- *Continuous-Time Fourier Transform (CTFT):*

$$\begin{aligned} X_c(j\Omega) &= \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt, \quad \Omega \in \mathbb{R} \\ x_c(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)e^{j\Omega t}d\Omega, \quad t \in \mathbb{R} \end{aligned}$$

  Note that:

  ♣ $X_c(j\Omega)$ is the Laplace transform evaluated at $s = j\Omega$.

  ♣ $x_c(t) \in \mathbb{R}$ implies $X_c(j\Omega) = X_c^*(-j\Omega)$, i.e., "conjugate symmetry".

  ♣ $x_c(t)$ is "bandlimited" to Ω_0 if $X_c(j\Omega) = 0$ for all $|\Omega| > \Omega_0$.

Using $x[n]$ to denote a discrete-time signal at index $n \in \mathbb{Z}$,

- *z-transform:*

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

  Denoting a transform pair by $x[n] \leftrightarrow X(z)$, some useful properties are

$$\begin{aligned} x[-n] &\leftrightarrow X(z^{-1}) \\ (-1)^n x[n] &\leftrightarrow X(-z) \end{aligned}$$

- *Discrete-Time Fourier Transform (DTFT):*

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \quad \omega \in \mathbb{R} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \quad n \in \mathbb{Z} \end{aligned}$$

Note that:

  - ♣ $X(e^{j\omega})$ is the z-transform evaluated on the unit circle in $\mathbb{C}$-plane: $z = e^{j\omega}$.
  - ♣ $X(e^{j\omega})$ is 2π-periodic in ω.
  - ♣ $x[n] \in \mathbb{R}$ implies $X(e^{j\omega}) = X^*(e^{-j\omega})$, i.e., "conjugate symmetry".

Other DTFT properties are:

$$\begin{aligned} x[-n] &\leftrightarrow X(e^{-j\omega}) \\ x^*[n] &\leftrightarrow X^*(e^{-j\omega}) \\ x[n-\ell] &\leftrightarrow X(e^{j\omega})e^{-j\omega\ell} \\ x[n]e^{j\omega_0 n} &\leftrightarrow X(e^{j(\omega-\omega_0)}) \\ x[n]y[n] &\leftrightarrow \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta \\ x[n]*y[n] &\leftrightarrow X(e^{j\omega})Y(e^{j\omega}) \\ \textstyle\sum_n |x[n]|^2 &\leftrightarrow \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \end{aligned}$$

where "$*$" denotes linear convolution: $x[n]*y[n] = \sum_{m=-\infty}^{\infty} x[m]\,y[n-m]$.

2 Uniform Sampling

Say that $x[n]$ is sampled from $x_c(t)$ with uniform sampling interval T:

$$\boxed{x[n] \;=\; x_c(nT)}, \quad n \in \mathbb{Z}$$

Let us define the continuous-time "impulse train"

$$p(t) \;=\; \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

where $\delta(t)$ denotes the Dirac delta function, defined by the properties

$$\begin{aligned} \textstyle\int_{-\infty}^{\infty} \delta(t)dt &= 1 && \text{"unit area"} \\ \textstyle\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt &= f(\tau) && \text{"sifting property"} \end{aligned}$$

Using Fourier series, it can be shown that

$$p(t) \;=\; \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{T}kt}$$

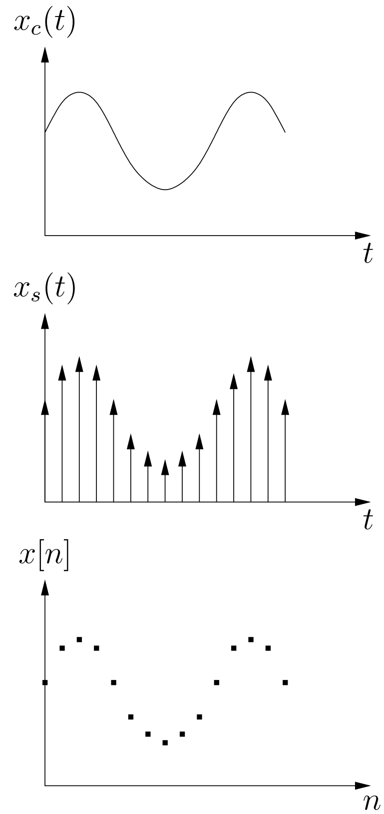

Figure 1: Signals used in sampling theorem.

Multiplying $x_c(t)$ by the impulse train yields the "continuous-time sampled signal" $x_s(t)$ which will help us to derive the sampling theorem. (See Fig. 1.)

$$\begin{aligned} x_s(t) &= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= x_c(t) \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{T}kt} \end{aligned}$$

Taking the CTFT of $x_s(t)$,

$$\begin{aligned} X_s(j\Omega) &= \int_{-\infty}^{\infty} \left(x_c(t) \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{T}kt} \right) e^{-j\Omega t} dt \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(t) e^{-j(\Omega - \frac{2\pi}{T}k)t} dt \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\big(j(\Omega - \tfrac{2\pi}{T}k)\big) \qquad (1) \end{aligned}$$

Notice that $X_s(j\Omega)$ is a summation of scaled and shifted copies of $X_c(j\Omega)$. When $X_c(j\Omega)$ is bandlimited to $\frac{\pi}{T}$ rad/s the spectral copies do not overlap, and we say there is no "aliasing." Aliasing may result when $X_c(j\Omega)$ is not bandlimited to $\frac{\pi}{T}$. (See Fig. 2.) The frequency $\frac{\pi}{T}$ rad/s, i.e., $\frac{1}{2T}$ Hz, is often called the "Nyquist frequency."

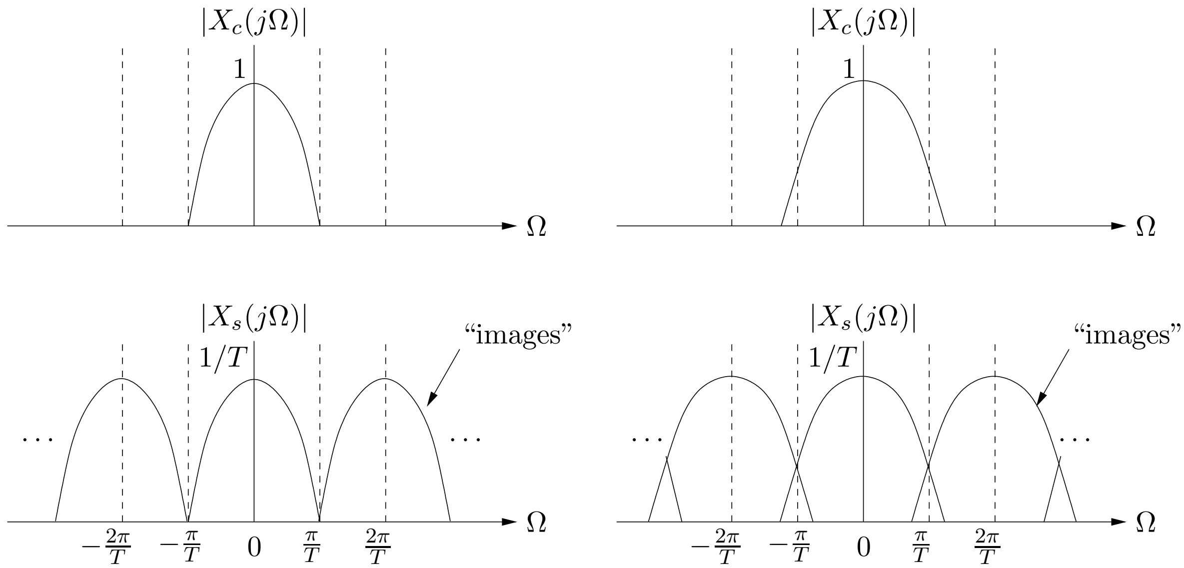

Figure 2: Example of no aliasing (left) and aliasing (right) in $X_s(j\Omega)$.

We now investigate the relationship between the CTFT and the DTFT:

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_c(t)\delta(t-nT)dt\right) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{x_c(t)\delta(t-nT)e^{-j\omega n}}_{\text{nonzero iff } n=t/T.} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(t)\delta(t-nT)e^{-j\omega\frac{t}{T}} dt \\
&= \int_{-\infty}^{\infty} \underbrace{x_c(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)}_{x_s(t)} e^{-j\frac{\omega}{T}t} dt \\
&= X_s\Big(j\frac{\omega}{T}\Big) \qquad (2)
\end{aligned}
$$

Plugging (1) into (2) yields

$$\boxed{X(e^{j\omega}) \;=\; \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\Big(j\Big(\frac{\omega-2\pi k}{T}\Big)\Big)}$$

Notice that the DTFT of the sampled signal $x[n]$ is a summation of scaled, stretched, and shifted copies of the CTFT of the continuous signal $x_c(t)$. As implied by (2), the DTFT may show evidence of aliasing when $x_c(t)$ is not bandlimited to the Nyquist frequency. (See Fig. 3.)

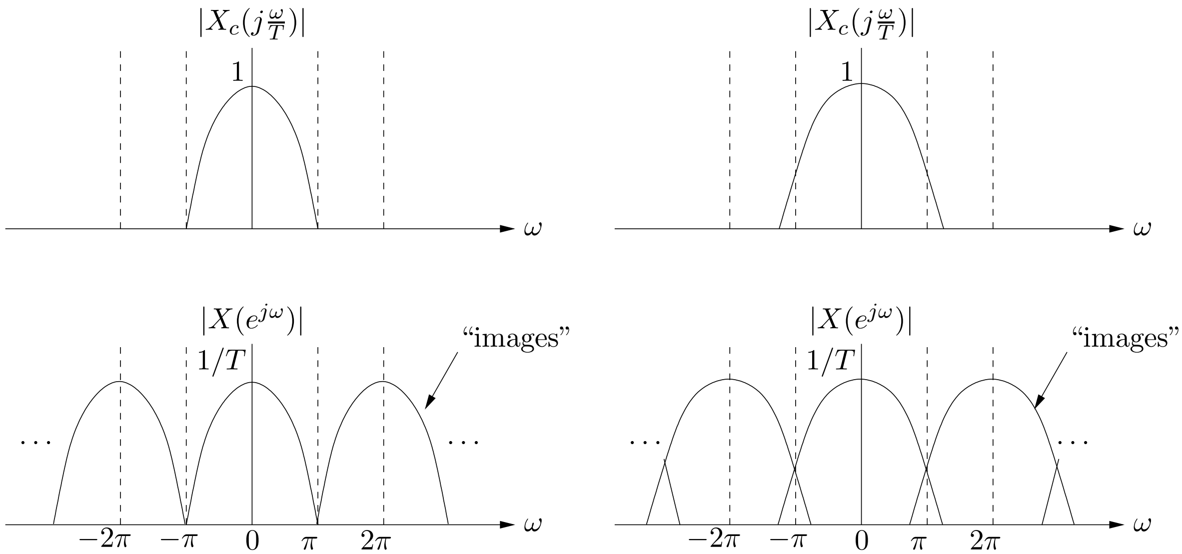

Figure 3: Example of no aliasing (left) and aliasing (right) in $X(e^{j\omega})$.

3 Reconstruction

Fig. 4 illustrates the theoretical procedure by which a Nyquist-bandlimited $x_c(t)$ can be perfectly reconstructed from its sampled representation $x[n]$. Essentially, it is a reversal of the sampling process. First, the discrete-time sequence $x[n]$ is convolved with a continuous-time Dirac delta to obtain $x_s(t)$.

$$x_s(t) \;=\; \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

As we know, $X_s(j\Omega)$ contains unwanted spectral copies, or images, and is scaled by a factor of $1/T$ relative to $X_c(j\Omega)$. Thus, the second step in the reconstruction procedure is to remove these images and scale by T. This can be accomplished by a "brick-wall" analog lowpass filter with cutoff at $\frac{\pi}{T}$ and DC gain of T. The time-domain impulse response of this filter is given by

$$h_b(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} H_b(j\Omega)e^{j\Omega t}d\Omega \;=\; \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} Te^{j\Omega t}d\Omega \;=\; \frac{\sin(\pi t/T)}{\pi t/T}$$

See Fig. 5 for an illustration. Thus, the reconstructed signal can be written

$$\boxed{x_c(t) \;=\; x_s(t) * h_b(t) \;=\; \sum_{n=-\infty}^{\infty} x[n]\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}} \tag{3}$$

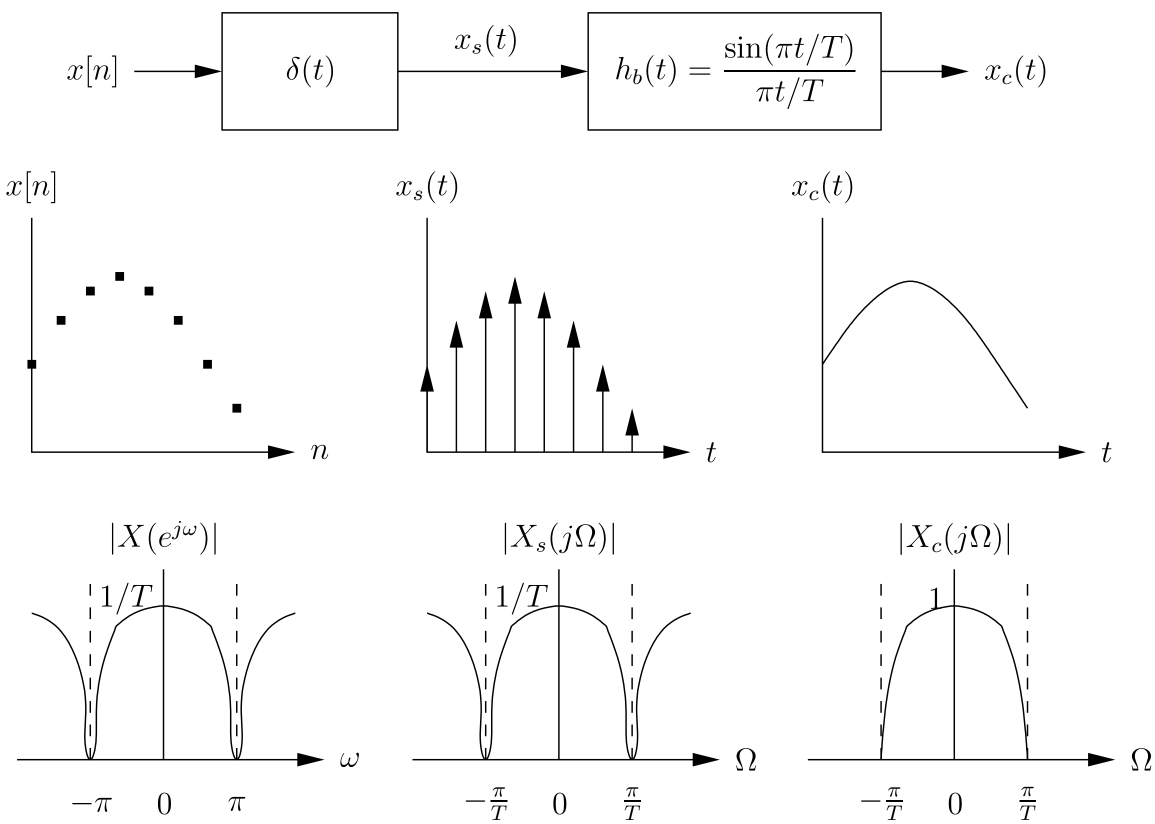

Figure 4: Ideal reconstruction of Nyquist-bandlimited signal.

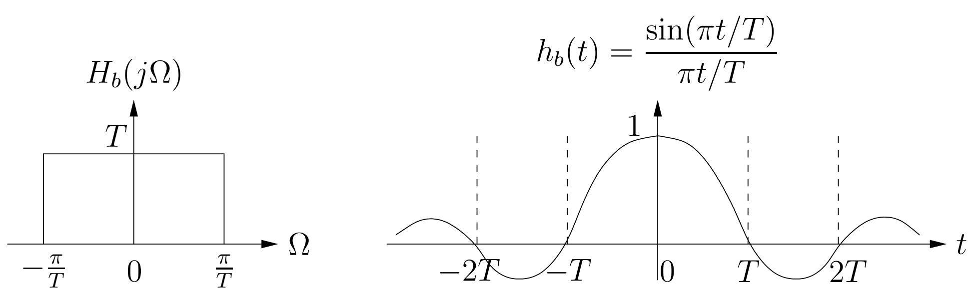

Figure 5: Brick-wall analog reconstruction lowpass filter.

In practice, it is not possible to generate Dirac deltas for the creation of $x_s(t)$. So, instead of convolving with $\delta(t)$, we might convolve with a rectangular pulse of width T, known as a zero-order hold (ZOH) function and denoted by $h_z(t)$. This yields $x_z(t)$:

$$x_z(t) = \sum_{n=-\infty}^{\infty} x[n] h_z(t - nT)$$

Unwanted spectral copies in $X_z(j\Omega)$ can be removed by a final stage of lowpass filtering. The two-step procedure is illustrated in Fig. 6. To be consistent with the Dirac delta, we assume a ZOH function with unit area, requiring that the analog reconstruction lowpass filter has DC gain T.

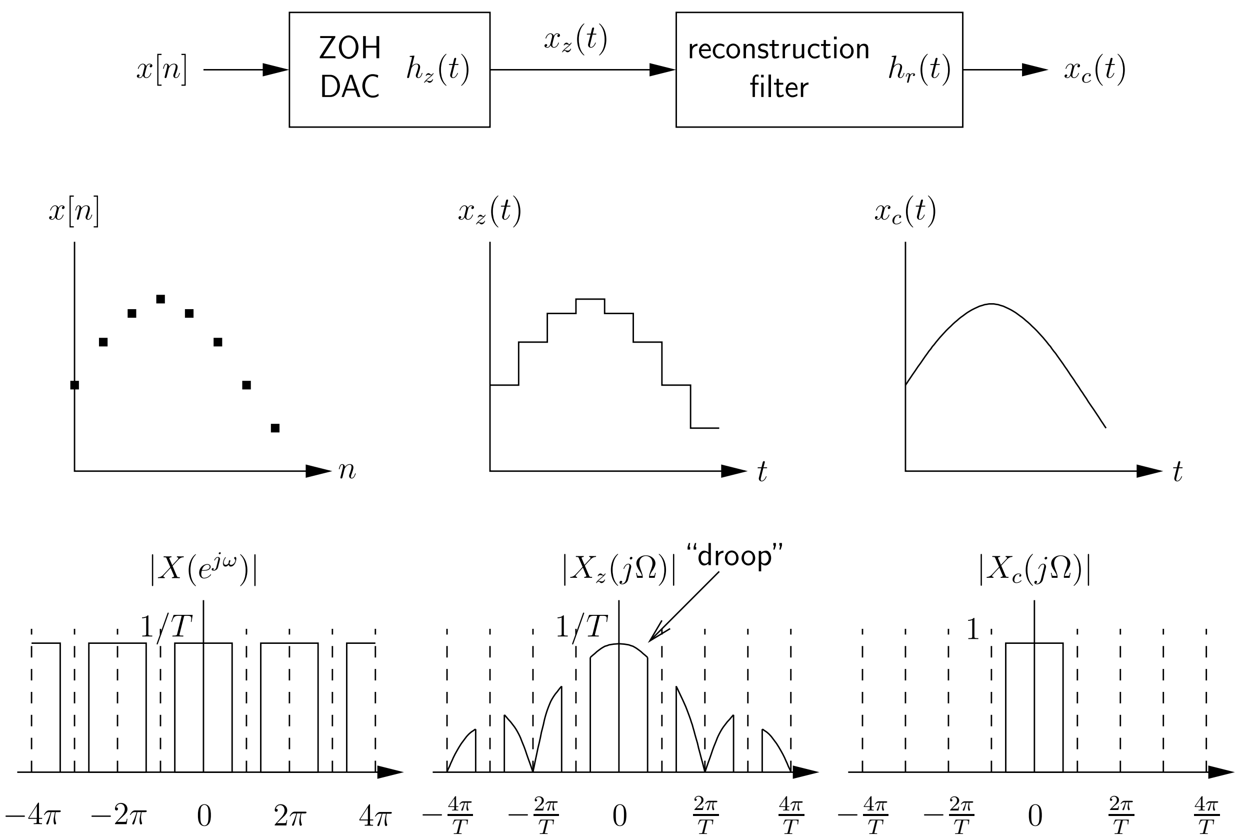

Figure 6: ZOH reconstruction of Nyquist-bandlimited signal.

Unlike convolution with $\delta(t)$, convolution with the ZOH function $h_z(t)$ introduces passband "droop" and out-of-band attenuation. This results from the fact that the frequency response magnitude of the ZOH function is not constant in Ω:

$$H_z(j\Omega) = \int_{-\infty}^{\infty} h_z(t) e^{-j\Omega t} dt = \int_0^T \frac{1}{T} e^{-j\Omega t} dt = \frac{\sin(\Omega T/2)}{\Omega T/2} e^{-j\Omega T/2}$$

(See Fig. 7 for an illustration.) Thus, for perfect reconstruction, the analog reconstruction filter $H_r(j\Omega)$ must invert the ZOH response $H_z(j\Omega)$ over the passband $[-\frac{\pi}{T}, \frac{\pi}{T})$. The left side of Fig. 8 shows the ideal $|H_r(j\Omega)|$.

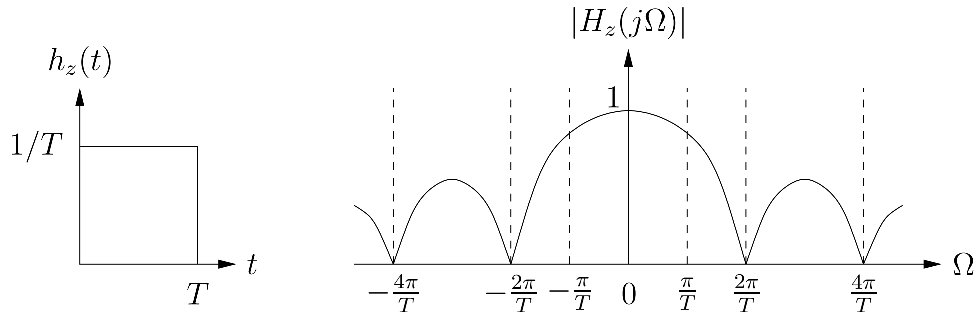

Figure 7: ZOH function.

It is difficult to build analog reconstruction filters with sharp cutoff. Instead, one hopes that the desired signal is bandlimited to less than the Nyquist frequency (as in Fig. 6), so that there is an absence of unwanted spectral energy in a region around $\pm\pi/T$. In this case, the analog reconstruction filter $H_r(j\Omega)$ may be designed with a wider transition band, such as on the right side of Fig. 8.

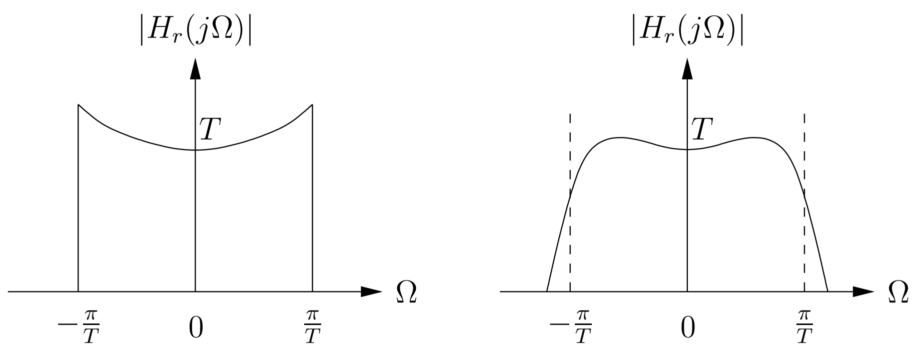

Figure 8: Ideal (left) and practical (right) reconstruction filters for ZOH.

4 Discrete Fourier Transform

N-point Discrete-Fourier Transform (DFT):

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \qquad k = 0 \dots N-1 \\ x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}, \qquad n = 0 \dots N-1 \end{aligned}$$

Note that:

- $X[k]$ is the DTFT evaluated at $\omega = \frac{2\pi}{N}k$ for $k = 0 \dots N-1$.
- Zero-padding $x[n]$ to M samples (where $M > N$) prior to an M-pt DFT yields an M-point uniformly sampled version of the DTFT:

$$X(e^{j\frac{2\pi}{M}k}) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}kn} = \sum_{n=0}^{M-1} x_{zp}[n] e^{-j\frac{2\pi}{M}kn} = X_{zp}[k], \quad k = 0 \dots M-1$$

  This can be used to compute a densely sampled DTFT of any N-pt sequence.

- An N-pt DFT can be interpolated to reconstruct the DTFT of an N-pt sequence.

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{-j\frac{2\pi}{N}kn} e^{-j\omega n} \\ &= \sum_{k=0}^{N-1} X[k] \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\left(\omega - \frac{2\pi}{N}k\right)n} \\ &= \sum_{k=0}^{N-1} X[k] \frac{1}{N}\frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} e^{-j\left(\omega - \frac{2\pi}{N}k\right)\frac{N-1}{2}} \end{aligned}$$

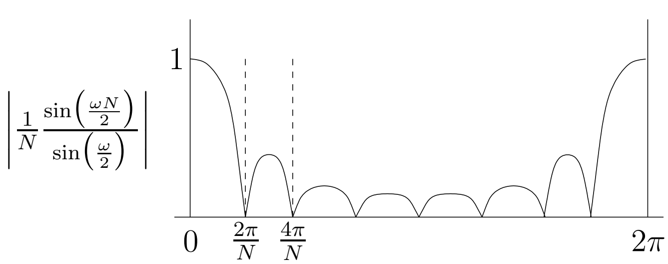

Figure 9: Dirichlet sinc

- The DFT has a convenient matrix representation.

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} e^{-j\frac{2\pi}{N}0} & e^{-j\frac{2\pi}{N}0} & e^{-j\frac{2\pi}{N}0} & e^{-j\frac{2\pi}{N}0} & \dots \\ e^{-j\frac{2\pi}{N}0} & e^{-j\frac{2\pi}{N}1} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}3} & \dots \\ e^{-j\frac{2\pi}{N}0} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & e^{-j\frac{2\pi}{N}6} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\mathbf{W}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\mathbf{x}}$$

  where $\mathbf{W}$ has the following properties:

  - $[\mathbf{W}]_{k,n} = e^{-j\frac{2\pi}{N}kn}$
  - $\mathbf{W}$ is symmetric, i.e., $\mathbf{W} = \mathbf{W}^t$.
  - $\frac{1}{\sqrt{N}}\mathbf{W}$ is unitary, i.e., $(\frac{1}{\sqrt{N}}\mathbf{W})(\frac{1}{\sqrt{N}}\mathbf{W})^H = (\frac{1}{\sqrt{N}}\mathbf{W})^H(\frac{1}{\sqrt{N}}\mathbf{W}) = \mathbf{I}$.
  - $\frac{1}{N}\mathbf{W}^* = \mathbf{W}^{-1}$, where $\mathbf{W}^{-1}$ is called the "IDFT matrix".
  - $\mathbf{W}$ is a "Vandermonde matrix", i.e., the n^{th} column of $\mathbf{W}$ is formed by raising $e^{-j\frac{2\pi}{N}n}$ to the powers $k = 0, 1, \dots, N-1$. The Vandermonde property implies that $\mathbf{W}$ is full rank.

- For $N = 2^m$ with $m \in \mathbb{N}$, the FFT algorithm can be used to compute the DFT using approximately $\frac{N}{2}\log_2 N$, rather than N^2, operations.

N	$\frac{N}{2}\log_2 N$	N^2	savings factor
16	32	256	8
64	192	4096	21.3
256	1024	65536	64
1024	5120	1048576	204.8

5 Miscellaneous

Linear Phase: A "linear-phase" filter $H(z)$ has a phase response that is linear in frequency. Specifically, the DTFT of a linear phase filter can be written as

$$H(e^{j\omega}) = e^{-j\omega d}\tilde{H}(e^{j\omega}), \quad \begin{cases} d \in \mathbb{R} \\ \tilde{H}(e^{j\omega}) \in \mathbb{R} \end{cases}$$

When $d = 0$, we refer to the filter as "zero-phase."

It can be shown that a filter is zero-phase iff its impulse response is conjugate symmetric around the origin, i.e., $h[m] = h^*[-m]\ \forall m \in \mathbb{N}$. (See the proof below.) Since symmetric length-N causal filters exhibit coefficient symmetry around the index $\frac{N-1}{2}$, the same arguments can be used to show that such filters are linear phase with $d = \frac{N-1}{2}$.

Proof. If $H(e^{j\omega})$ is real-valued, then following must hold $\forall\omega$:

$$\begin{aligned}
0 &= \mathrm{Im}\left\{\sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega}\right\} \\
&= \mathrm{Im}\left\{\sum_{n=-\infty}^{\infty} \big(h_r[n]+jh_i[n]\big)\big(\cos(n\omega)-j\sin(n\omega)\big)\right\} \\
&= \sum_{n=-\infty}^{\infty} h_i[n]\cos(n\omega)-h_r[n]\sin(n\omega) \\
&= h_i[0]+\sum_{n=1}^{\infty}\big(h_i[n]+h_i[-n]\big)\cos(n\omega)-\big(h_r[n]-h_r[-n]\big)\sin(n\omega)
\end{aligned}$$

If $f(\omega)=0\ \forall\omega$, then

$$0 \;=\; \lim_{\omega_0\to\infty}\frac{2}{\omega_0}\int_{-\omega_0}^{\omega_0} f(\omega)\sin(\omega m)d\omega \quad \forall m\in\mathbb{N}$$

Using $f(\omega)=\mathrm{Im}\{H(e^{j\omega})\}$, we find that

$$0 \;=\; h_r[m]-h_r[-m] \quad \forall m\in\mathbb{N}.$$

Similarly, if $f(\omega)=0\ \forall\omega$, then

$$0 \;=\; \lim_{\omega_0\to\infty}\frac{2}{\omega_0}\int_{-\omega_0}^{\omega_0} f(\omega)\cos(\omega m)d\omega \quad \forall m\in\mathbb{N}$$

Using $f(\omega)=\mathrm{Im}\{H(e^{j\omega})\}$, we find that

$$0 \;=\; h_i[m]+h_i[-m] \quad \forall m\in\mathbb{Z}^+.$$

Putting the real/imaginary coefficient requirements together, we have

$$h[m]=h^*[-m] \quad \forall m\in\mathbb{Z}^+.$$

Thus we have shown that a real-valued DTFT implies conjugate symmetric coefficients (with symmetry around the origin).

To prove the other direction, assume $h[m]=h^*[-m]\ \ \forall m\in\mathbb{Z}^+$. Then

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega} \\
&= h[0]+\sum_{n=1}^{\infty} h[n]e^{-jn\omega}+h[-n]e^{jn\omega} \\
&= h[0]+\sum_{n=1}^{\infty}(h[n]e^{-jn\omega})+(h[n]e^{-jn\omega})^* \\
&= h[0]+2\mathrm{Re}\left\{\sum_{n=1}^{\infty} h[n]e^{-jn\omega}\right\} \\
&\in \mathbb{R}\ \ \forall\omega
\end{aligned}$$

since the conjugate symmetry implies that $h[0]\in\mathbb{R}$. Thus we have shown that conjugate symmetric coefficients (with symmetry around the origin) imply a real-valued DTFT. □

Group-Delay Response: The group-delay response of a discrete-time linear system $H(z)$ is defined as the negative derivative of the phase response of the DTFT $H(e^{jw})$. In other words, if we write

$$H(e^{j\omega}) \;=\; |H(e^{j\omega})|e^{j\phi(\omega)}$$

where $\phi(\omega)$ denotes the phase response, then the group-delay response $g(\omega)$ is

$$\boxed{g(\omega) \;=\; -\frac{\partial\phi(\omega)}{\partial\omega}}$$

Functionally, $g(\omega)$ describes the delay (in samples) imposed by $H(z)$ on input signal components with frequency ω.

Recall that the DTFT of a symmetric causal length-N filter can be written in terms of real-valued $\tilde{H}(e^{j\omega})$ as

$$H(e^{j\omega}) \;=\; \tilde{H}(e^{j\omega})e^{-j\frac{N-1}{2}\omega}$$

from which it is evident that $|H(e^{j\omega})| = |\tilde{H}(e^{j\omega})| = \operatorname{sgn}\{\tilde{H}(e^{j\omega})\} \cdot \tilde{H}(e^{j\omega})$. Thus

$$H(e^{j\omega}) \;=\; |H(e^{j\omega})| \cdot \operatorname{sgn}\{\tilde{H}(e^{j\omega})\}e^{-j\frac{N-1}{2}\omega}$$

which implies that

$$\begin{aligned}
\phi(\omega) &\;=\; \begin{cases} -\frac{N-1}{2}\omega & \omega \text{ s.t. } \operatorname{sgn}\{\tilde{H}(e^{j\omega})\} = 1 \\ -\frac{N-1}{2}\omega + \pi & \omega \text{ s.t. } \operatorname{sgn}\{\tilde{H}(e^{j\omega})\} = -1 \end{cases} \\
g(\omega) &\;=\; \begin{cases} \frac{N-1}{2} & \omega \text{ s.t. } H(e^{j\omega}) \neq 0 \\ \text{undefined} & \omega \text{ s.t. } H(e^{j\omega}) = 0 \end{cases}
\end{aligned}$$

Note that this group-delay response is constant with respect to frequency, implying that all input components not completely attenuated by the system will be delayed by the same amount.