EE-597 Class Notes – DPCM

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1 DPCM

1.1 Pulse Code Modulation

- PCM is the "standard" sampling method taught in introductory DSP courses. The input signal is
  1. filtered to prevent aliasing,
  2. discretized in time by a sampler, and
  3. quantized in amplitude by a quantizer

  before transmission (or storage). Finally, the received samples are interpolated in time and amplitude to reconstruct an approximation of the input signal. Note that transmission may employ the use of additional encoding and decoding, as with entropy codes.

  PCM is the most widespread and well-understood digital coding system for reasons of simplicity, though not efficiency (as we shall see).

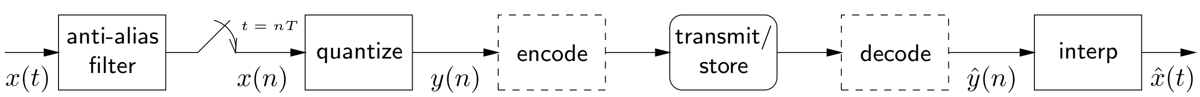

Figure 1: Standard PCM system.

1.2 Differential Pulse Code Modulation

- Many information signals, including audio, exhibit significant redundancy between successive samples. In these situations, it is advantageous to transmit only the difference between predicted and true versions of the signal: with a "good" predictions, the quantizer input will have variance less than the original signal, allowing a quantizer with smaller decision regions and hence higher SNR. (See Fig. 4 for an example of such a structure.)

- <u>*Linear Prediction:*</u> There are various methods of prediction, but we focus on *forward linear prediction of order N*, illustrated by Fig. 2 and described by the following equation, where $\hat{x}(n)$ is a linear estimate of $x(n)$ based on N previous versions of $x(n)$:

$$\hat{x}(n) = \sum_{i=1}^{N} h_i x(n-i). \tag{1}$$

  It will be convenient to collect the *prediction coefficients* into the vector $\mathbf{h} = (h_1, h_2, \ldots, h_N)^t$.

- <u>*Lossless Predictive Encoding:*</u> Consider first the system in Fig. 3. The system equations are

$$\begin{aligned} e(n) &= x(n) - \hat{x}(n) \\ y(n) &= e(n) + \hat{y}(n). \end{aligned}$$

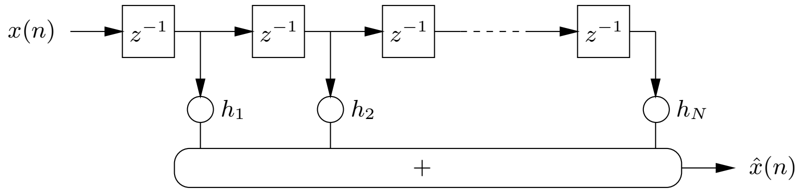

Figure 2: Linear Prediction.

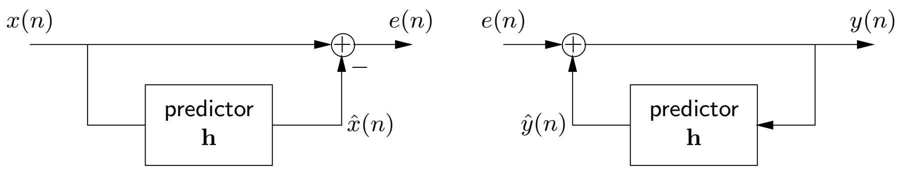

Figure 3: Lossless Predictive Data Transmission System.

In the z-domain (i.e., $X(z) = \sum_n x(n) z^{-n}$ and $H(z) = \sum_i h_i z^{-i}$),

$$\begin{aligned} E(z) &= X(z) - \hat{X}(z) = X(z)\big(1 - H(z)\big) \\ Y(z) &= E(z) + \hat{Y}(z) = E(z) + H(z)Y(z). \end{aligned}$$

We call this transmission system lossless because, from above,

$$Y(z) = \frac{E(z)}{1 - H(z)} = X(z).$$

Without quantization, however, the prediction error $e(n)$ takes on a continuous range of values, and so this scheme is not applicable to digital transmission.

- *Quantized Predictive Encoding:* Quantizing the prediction error in Fig. 3, we get the system of Fig. 4. Here the equations are

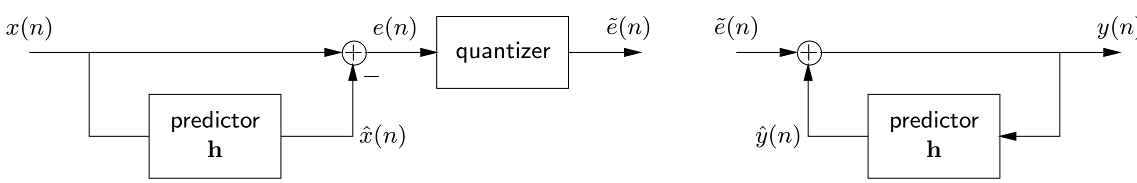

Figure 4: Quantized Predictive Coding System.

$$\begin{aligned} q(n) &= \tilde{e}(n) - e(n) \\ e(n) &= x(n) - \hat{x}(n) \\ y(n) &= \tilde{e}(n) + \hat{y}(n). \end{aligned}$$

In the z-domain we find that

$$\begin{aligned} \tilde{E}(z) &= X(z)\big(1 - H(z)\big) + Q(z) \\ Y(z) &= \frac{\tilde{E}(z)}{1 - H(z)} = X(z) + \frac{Q(z)}{1 - H(z)}. \end{aligned}$$

Thus the reconstructed output is corrupted by a filtered version of the quantization error where the filter $\big(1 - H(z)\big)^{-1}$ is expected to amplify the quantization error; recall that $Y(z) = E(z)\big(1 - H(z)\big)^{-1}$ where the goal of prediction was to make $\sigma_e^2 < \sigma_y^2$. This problem results from the fact that the quantization noise appears at the decoder's predictor input but not at the encoder's predictor input. But we can avoid this...

- <u>*DPCM:*</u> Including quantization in the encoder's prediction loop, we obtain the system in Fig. 5, known as *differential pulse code modulation*. System equations are

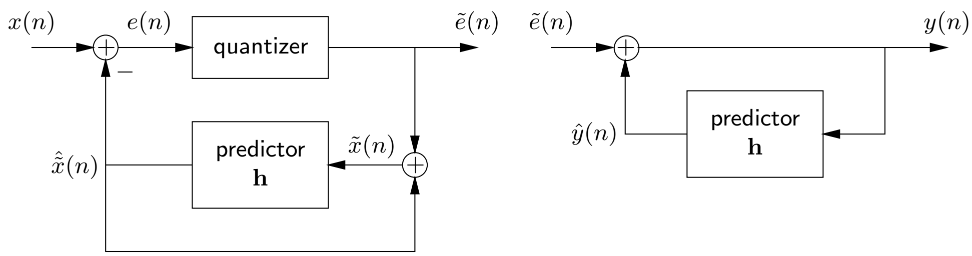

Figure 5: A Typical Differential PCM System.

$$
\begin{aligned}
q(n) &= \tilde{e}(n) - e(n) \\
e(n) &= x(n) - \hat{\tilde{x}}(n) \\
\tilde{x}(n) &= \tilde{e}(n) + \hat{\tilde{x}}(n) \\
y(n) &= \tilde{e}(n) + \hat{y}(n).
\end{aligned}
$$

In the z-domain we find that

$$
\begin{aligned}
\tilde{E}(z) &= X(z) - H(z)\tilde{X}(z) + Q(z) \\
\tilde{X}(z) &= \big(Q(z) + E(z)\big) + \big(X(z) - E(z)\big) \;=\; X(z) + Q(z) \\
Y(z) &= \frac{\tilde{E}(z)}{1 - H(z)}
\end{aligned}
$$

so that

$$
Y(z) = \frac{X(z) - H(z)\big(X(z) + Q(z)\big) + Q(z)}{1 - H(z)} \;=\; X(z) + Q(z) \;=\; \tilde{X}(z).
$$

Thus, the reconstructed output is corrupted only by the quantization error.

Another significant advantage to placing the quantizer inside the prediction loop is realized if the predictor made self-adaptive (in the same spirit as the adaptive quantizers we studied). As illustrated in Fig. 6, adaptation of the prediction coefficients can take place simulateously at the encoder and decoder with no transmission of side-information (e.g. $\mathbf{h}(n)$)! This is a consequence of the fact that both algorithms have access to identical signals.

1.3 Performance of DPCM

- As we noted earlier, the DPCM performance gain is a consequence of variance reduction obtained through prediction. Here we derive the optimal predictor coefficients, prediction error variance, and bit rate for the system in Fig. 4. This system is easier to analyze than DPCM systems with quantizer in loop (e.g., Fig. 5) and it is said that the difference in prediction-error behavior is negligible when $R > 2$ [2, p. 267].

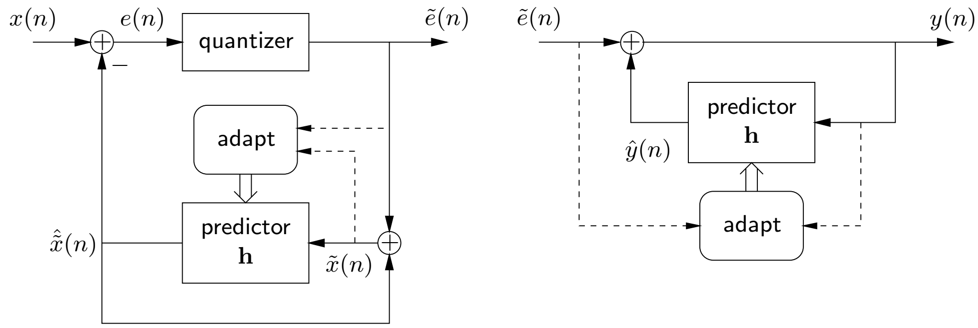

Figure 6: Adaptive DPCM System.

- *Optimal Prediction Coefficients:* First we find coefficients $\mathbf{h}$ minimizing prediction error variance:

$$\min_{\mathbf{h}} \mathrm{E}\{e^2(n)\}.$$

Throughout, we assume that $x(n)$ is a zero-mean stationary random process with autocorrelation

$$r_x(k) := \mathrm{E}\{x(n)x(n-k)\} = r_x(-k).$$

A necessary condition for optimality is the following:

$$\begin{aligned}
\forall j \in \{1, \dots, N\}, \quad 0 &= \frac{1}{2}\frac{\partial}{\partial h_j}\mathrm{E}\{e^2(n)\} \\
&= \mathrm{E}\left\{e(n)\frac{\partial e(n)}{\partial h_j}\right\} \\
&= \mathrm{E}\left\{e(n)x(n-j)\right\} \qquad \leftarrow \text{The "Orthogonality Principle"} \\
&= \mathrm{E}\left\{\left(x(n) - \sum_{i=1}^{N} h_i x(n-i)\right)x(n-j)\right\} \\
&= \mathrm{E}\left\{x(n)x(n-j)\right\} - \sum_{i=1}^{N} h_i \,\mathrm{E}\left\{x(n-i)x(n-j)\right\} \\
&= r_x(j) - \sum_{i=1}^{N} h_i r_x(j-i)
\end{aligned}$$

where we have used (1). We can rewrite this as a system of linear equations:

$$\underbrace{\begin{pmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(N) \end{pmatrix}}_{\mathbf{r}_x} = \underbrace{\begin{pmatrix} r_x(0) & r_x(1) & \dots & r_x(N-1) \\ r_x(1) & r_x(0) & \dots & r_x(N-2) \\ \vdots & & & \\ r_x(N-1) & r_x(N-2) & \dots & r_x(0) \end{pmatrix}}_{\mathbf{R}_N} \underbrace{\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix}}_{\mathbf{h}}$$

which yields an expression for the optimal prediction coefficients:

$$\boxed{\mathbf{h}_\star = \mathbf{R}_N^{-1}\mathbf{r}_x}. \qquad (2)$$

- *Error for Length-N Predictor:* The definition $\mathbf{x}(n) := \big(x(n), x(n-1), \dots, x(n-N)\big)^t$ and (2) can be used to show that the minimum prediction error variance is

$$\begin{aligned}
\sigma_e^2\big|_{\min,N} &= \mathrm{E}\{e^2(n)\} \\
&= \mathrm{E}\left\{\left\|\mathbf{x}^t(n)\begin{pmatrix}1\\-\mathbf{h}_\star\end{pmatrix}\right\|^2\right\} \\
&= \begin{pmatrix}1 & -\mathbf{h}_\star^t\end{pmatrix}\mathrm{E}\{\mathbf{x}(n)\mathbf{x}^t(n)\}\begin{pmatrix}1\\-\mathbf{h}_\star\end{pmatrix} \\
&= \begin{pmatrix}1 & -\mathbf{h}_\star^t\end{pmatrix}\begin{pmatrix}r_x(0) & \mathbf{r}_x^t\\ \mathbf{r}_x & \mathbf{R}_N\end{pmatrix}\begin{pmatrix}1\\-\mathbf{h}_\star\end{pmatrix} \\
&= r_x(0) - 2\mathbf{h}_\star^t\mathbf{r}_x + \mathbf{h}_\star^t\mathbf{R}_N\mathbf{h}_\star \\
&= r_x(0) - \mathbf{r}_x^t\mathbf{R}_N^{-1}\mathbf{r}_x.
\end{aligned}$$

- <u>*Error for Infinite-Length Predictor:*</u> We now characterize $\sigma_e^2\big|_{\min,N}$ as $N\to\infty$. Note that

$$\underbrace{\begin{pmatrix}r_x(0) & \mathbf{r}_x^t\\ \mathbf{r}_x & \mathbf{R}_N\end{pmatrix}}_{\mathbf{R}_{N+1}}\begin{pmatrix}1\\-\mathbf{h}_\star\end{pmatrix} = \begin{pmatrix}\sigma_e^2\big|_{\min,N}\\ \mathbf{0}\end{pmatrix}$$

Using Cramer's rule,

$$1 = \frac{\begin{vmatrix}\sigma_e^2\big|_{\min,N} & \mathbf{r}_x^t\\ \mathbf{0} & \mathbf{R}_N\end{vmatrix}}{|\mathbf{R}_{N+1}|} = \sigma_e^2\big|_{\min,N}\frac{|\mathbf{R}_N|}{|\mathbf{R}_{N+1}|} \quad\Rightarrow\quad \boxed{\sigma_e^2\big|_{\min,N} = \frac{|\mathbf{R}_{N+1}|}{|\mathbf{R}_N|}}.$$

Aside 1.1 (Cramer's Rule):
Given matrix equation $\mathbf{Ay}=\mathbf{b}$, where $\mathbf{A}=(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_N)\in\mathbb{R}^{N\times N}$,

$$y_k = \frac{\big|\mathbf{a}_1,\ldots,\mathbf{a}_{k-1},\mathbf{b},\mathbf{a}_{k+1},\ldots,\mathbf{a}_N\big|}{\big|\mathbf{A}\big|}$$

where $|\cdot|$ denotes determinant.

A result from the theory of Toeplitz determinants [2] gives the final answer:

$$\sigma_e^2\big|_{\min} = \lim_{N\to\infty}\frac{|\mathbf{R}_{N+1}|}{|\mathbf{R}_N|} = \boxed{\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln S_x(e^{j\omega})d\omega\right)} \tag{3}$$

where $S_x(e^{j\omega})$ is the *power spectral density* of the WSS random process $x(n)$:

$$S_x(e^{j\omega}) := \sum_{n=-\infty}^{\infty} r_x(n)e^{-j\omega n}.$$

(Note that, because $r_x(n)$ is conjugate symmetric for stationary $x(n)$, $S_x(e^{j\omega})$ will always be non-negative and real.)

- <u>*ARMA Source Model:*</u> If the random process $x(n)$ can be modelled as a *general linear process*, i.e., white noise $v(n)$ driving a causal LTI system $B(z)$:

$$x(n) = v(n) + \sum_{k=1}^{\infty} b_k v(n-k) \quad \text{with} \quad \sum_k |b_k|^2 < \infty,$$

then it can be shown that

$$\sigma_v^2 = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_x(e^{j\omega})d\omega\right).$$

Thus the MSE-optimal prediction error variance equals that of the driving noise $v(n)$ when $N = \infty$.

- *Prediction Error Whiteness:* We can also demonstrate that the MSE-optimal prediction error is white when $N = \infty$. This is a simple fact of the orthogonality principle seen earlier:

$$\boxed{0 = \mathrm{E}\{e(n)x(n-k)\}, \quad k = 1, 2, \dots}.$$

The prediction error has autocorrelation

$$\begin{aligned}
\mathrm{E}\{e(n)e(n-k)\} &= \mathrm{E}\left\{e(n)\left(x(n-k) + \sum_{i=1}^{\infty} h_i x(n-k-i)\right)\right\} \\
&= \underbrace{\mathrm{E}\{e(n)x(n-k)\}}_{\to 0 \text{ for } k>0} + \sum_{i=1}^{\infty} h_i \underbrace{\mathrm{E}\{e(n)x(n-k-i)\}}_{\to 0} \\
&= \sigma_e^2\big|_{\min}\delta(k).
\end{aligned}$$

- *AR Source Model:* When the input can be modelled as an autoregressive (AR) process of order N:

$$X(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} V(z),$$

then MSE-optimal results (i.e., $\sigma_e^2 = \sigma_e^2\big|_{\min}$ and whitening) may be obtained with a forward predictor of order N. Specifically, the prediction coefficients h_i can be chosen as $h_i = a_i$ and so the prediction error $E(z)$ becomes

$$E(z) = \big(1 - H(z)\big)X(z) = \frac{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} V(z) = V(z),$$

Thus $\sigma_e^2 = \sigma_v^2 = \sigma_e^2\big|_{\min}$, and white $V(z)$ implies white $E(z)$.

- *Efficiency Gain over PCM:* Prediction reduces the variance at the quantizer input without changing the variance of the reconstructed signal.
  - By keeping the number of quantization levels fixed, could reduce quantization step width and obtain lower quantization error than PCM at the same bit rate.
  - By keeping the decision levels fixed, could reduce the number of quantization levels and obtain a lower bit rate than PCM at the same quantization error level.

Assuming that $x(n)$ and $e(n)$ are distributed similarly, use of the same style of quantizer on DPCM vs. PCM systems yields

$$\boxed{\mathrm{SNR}_{\mathrm{DPCM}} = \mathrm{SNR}_{\mathrm{PCM}} + 10\log_{10}\frac{\sigma_x^2}{\sigma_e^2}}.$$

1.4 Connections to Rate-Distortion Theory

- The *rate-distortion function* $R(D)$ specifies the minimum average rate R required to transmit the source process at a mean distortion of D, while the *distortion-rate function* $D(R)$ specifies the minimum mean distortion D resulting from transmission of the source at average rate R. These bounds are theoretical in the sense that coding techniques which attain these minimum rates or distortions are in general unknown and thought to be infinitely complex as well as require infinite memory. Still, these bounds form a reference against which any specific coding system can be compared.

For a continuous-amplitude white (i.e., "memoryless") Gaussian source $x(n)$ [1, 2],

$$\begin{aligned} R(D) &= \begin{cases} \frac{1}{2}\log_2 \frac{\sigma_x^2}{D} & 0 \le D \le \sigma_x^2 \\ 0 & D \ge \sigma_x^2 \end{cases} \\ D(R) &= 2^{-2R}\sigma_x^2. \end{aligned}$$

The sources we are interested in, however, are non-white. It turns out that when distortion D is "small," non-white Gaussian $x(n)$ have the following distortion-rate function [2, p. 644]:

$$\begin{aligned} D(R) &= 2^{-2R}\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln S_x(e^{j\omega})d\omega\right) \\ &= 2^{-2R}\sigma_x^2 \underbrace{\left(\frac{\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln S_x(e^{j\omega})d\omega\right)}{\frac{1}{2\pi}\int_{-\pi}^{\pi}S_x(e^{j\omega})d\omega}\right)}_{\text{spectral flatness measure}}. \end{aligned}$$

Note the ratio of geometric to arithmetic PSD means, called the *spectral flatness measure.* Thus optimal coding of $x(n)$ yields

$$\begin{aligned} \mathrm{SNR}(R) &= 10\log_{10}\left(\frac{\sigma_x^2}{D(R)}\right) \\ &\approx \boxed{6.02R - 10\log_{10}(\mathrm{SFM}_x)}. \end{aligned} \tag{4}$$

To summarize, (4) gives the best possible SNR for *any* arbitrarily-complex coding system that transmits/stores information at an average rate of R bits/sample.

- Let's compare the SNR-versus-rate performance acheivable by DPCM to the optimal given by (4). The structure we consider is shown in Fig. 7, where quantized DPCM outputs $\tilde{e}(n)$ are coded into binary bits using an entropy coder. Assuming that $\tilde{e}(n)$ is white (which is a good assumption for well-designed predictors), optimal entropy coding/decoding is able to transmit and recover $\tilde{e}(n)$ at $R = H_{\tilde{e}}$ bits/sample without any distortion. $H_{\tilde{e}}$ is the entropy of $\tilde{e}(n)$, for which we derived the following expression assuming large-L uniform quantizer:

$$H_{\tilde{e}} = h_e - \frac{1}{2}\log_2\big(12\mathrm{var}(e(n) - \tilde{e}(n))\big).$$

Since $\mathrm{var}(e(n) - \tilde{e}(n)) = \sigma_r^2$ in DPCM, $H_{\tilde{e}}$ can be rewritten:

$$H_{\tilde{e}} = h_e - \frac{1}{2}\log_2\big(12\sigma_r^2\big).$$

If $e(n)$ is Gaussian, it can be shown that the differential entropy h_e takes on the value

$$h_e = \frac{1}{2}\log_2\big(2\pi e\sigma_e^2\big),$$

so that

$$H_{\tilde{e}} = \frac{1}{2}\log_2\big(\frac{\pi e\sigma_e^2}{6\sigma_r^2}\big).$$

Using $R = H_{\tilde{e}}$ and rearranging the previous expression, we find

$$\sigma_r^2 = \frac{\pi e}{6}2^{-2R}\sigma_e^2.$$

With the optimal infinite length predictor, σ_e^2 equals $\sigma_e^2\big|_{\min}$ given by (3). Plugging (3) into the previous expression and writing the result in terms of the spectral flatness measure,

$$\sigma_r^2 = \frac{\pi e}{6}2^{-2R}\sigma_x^2\,\mathrm{SFM}_x.$$

Translating into SNR, we obtain

$$\begin{aligned} \text{SNR} &= 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_r^2}\right) \\ &\approx \boxed{6.02R - 1.53 - 10\log_{10}\text{SFM}_x} \quad [\text{dB}]. \end{aligned} \tag{5}$$

To summarize, a DPCM system using a MSE-optimal infinite-length predictor and optimal entropy coding of $\tilde{e}(n)$ could operate at an average of R bits/sample with the SNR in (5).

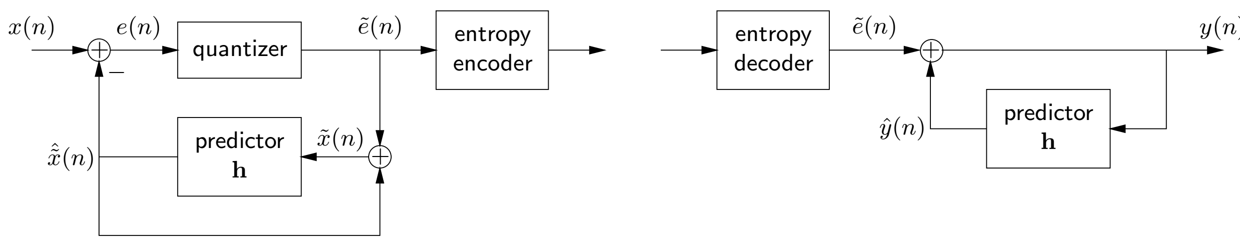

Figure 7: Entropy-Encoded DPCM System.

- Comparing (4) and (5), we see that DPCM incurs a 1.5 dB penalty in SNR when compared to the optimal. From our previous discussion on optimal quantization, we recognize that this 1.5 dB penalty comes from the fact that the quantizer in the DPCM system is memoryless. (Note that the DPCM quantizer *must* be memoryless since the predictor input must not be delayed.)
- Though we have identified a 1.5 dB DPCM penalty with respect to optimal, a key point to keep in mind is that the design of near-optimal coders for *non-white* signals is extremely difficult. When the signal statistics are rapidly changing, such a design task becomes nearly impossible.

  Though still non-trivial to design, near-optimal entropy coders for *white* signals exist and are widely used in practice. Thus, DPCM can be thought of as a way of pre-processing a colored signal that makes near-optimal coding possible. From this viewpoint, 1.5 dB might not be considered a high price to pay.

References

[1] T. Berger, *Rate Distortion Theory*, Englewood Cliffs, NJ: Prentice-Hall, 1971.

[2] N.S. Jayant and P. Noll, *Digital Coding of Waveforms*, Englewood Cliffs, NJ: Prentice-Hall, 1984.