MIDTERM #1 SOLUTIONS

1. (a) For a path length of d and propagation speed c, the propagation delay equals $\tau = d/c$. So, the direct path delay is $\boxed{\tau_d = \frac{3\times10^3\text{m}}{3\times10^8\text{m/sec}} = 1\times10^{-5}\text{sec} = 10\mu\text{sec}}$, and the indirect path delay is $\boxed{\tau_i = \frac{6\times10^3\text{m}}{3\times10^8\text{m/sec}} = 2\times10^{-5}\text{sec} = 20\mu\text{sec}}$.

   (b) Here we model the channel that produces $x(t)$ from $s(t)$ as a linear system with impulse response $h(t)$. In other words, $x(t) = h(t)*s(t)$. Because $x(t)$ is a superposition of two delayed versions of $s(t)$, one scaled by $\frac{1}{2}$, we can write

   $$x(t) = s(t-\tau_d) + \tfrac{1}{2}s(t-\tau_i).$$

   Now, if $s(t)$ had been the impulse $\delta(t)$, then the output would have been $\delta(t-\tau_d) + \frac{1}{2}\delta(t-\tau_i)$. Thus, the "impulse response" equals

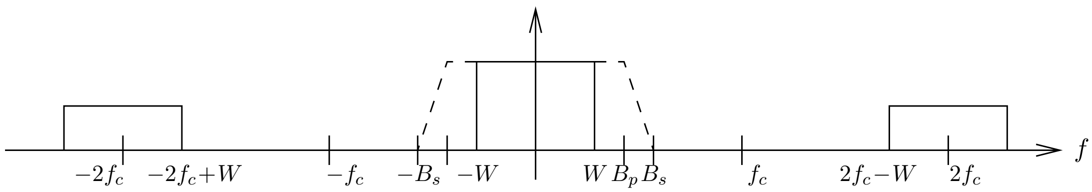
   $$\boxed{h(t) = \delta(t-\tau_d) + \tfrac{1}{2}\delta(t-\tau_i).}$$

   (c) Notice that, for sampling rate $\frac{1}{T_s} = 100$ kHz, the sampling interval is $T_s = 1\times10^{-5}\text{sec} = 10\mu\text{sec}$. Thus, τ_d corresponds to 1 sample of delay, while τ_i corresponds to 2 samples of delay. The causal sampled impulse response vector that we would pass to `plottf` would then be

   $$\boxed{\texttt{h} = \left[0, \tfrac{1}{T_s}, \tfrac{0.5}{T_s}\right] = [0, 100000, 50000].}$$

   Recall, from the lecture and the homework, that a Dirac delta is approximated in discrete-time using a single spike of height $\frac{1}{T_s}$.

2. (a) When $\theta = 0 = \phi$, we have synchronized AM modulation and demodulation. For this case, the figure below illustrates the spectrum at the output of the cosine multiplier in the demodulator in solid lines, and the LPF magnitude response in dashed lines.



For perfect demodulation, we need to preserve the signal up to W Hz and suppress it after $2f_c - W$ Hz. Thus, we need $\boxed{B_p \geq W}$ and $\boxed{B_s \leq 2f_c - W}$.

(b) For general θ and ϕ, we have

$$\begin{aligned} v(t) &= \mathsf{LPF}\{r(t)\, 2\cos(2\pi f_c t + \phi)\} && (1)\\ &= \mathsf{LPF}\{m(t)\, \underbrace{2\cos(2\pi f_c t + \theta)\cos(2\pi f_c t + \phi)}_{\cos(4\pi f_c t + \theta + \phi) + \cos(\theta - \phi)}\} && (2)\\ &= \mathsf{LPF}\{m(t)\cos(4\pi f_c t + \theta + \phi) + m(t)\cos(\theta - \phi)\} && (3)\\ &= m(t)\cos(\theta - \phi) && (4) \end{aligned}$$

(c) For general θ and ϕ, we have

$$\begin{aligned} c(t) &= \mathsf{LPF}\{r(t)\, 2\sin(2\pi f_c t + \phi)\} && (5)\\ &= \mathsf{LPF}\{m(t)\, \underbrace{2\cos(2\pi f_c t + \theta)\sin(2\pi f_c t + \phi)}_{\sin(4\pi f_c t + \theta + \phi) + \sin(\phi - \theta)}\} && (6)\\ &= \mathsf{LPF}\{m(t)\sin(4\pi f_c t + \theta + \phi) + m(t)\sin(\phi - \theta)\} && (7)\\ &= m(t)\sin(\phi - \theta), && (8) \end{aligned}$$

where lowpass filtering suppressed the double-frequency term $\sin(4\pi f_c t + \theta + \phi)$.

(d) From the answer to part (b), we see that setting $\boxed{\phi = \theta}$ gives $v(t) = m(t)$, i.e., perfect demodulation. But even if you didn't get a clear answer to part (b), the block diagram alone suggests that $\phi = \theta$ would yield a phase-synchronous AM system, which we know yields perfect demodulation.

(e) Notice that under the perfect-demod setting $\phi = \theta$, the control signal becomes $\boxed{c(t) = 0}$. Thus, even if we didn't know θ directly, we could still adjust ϕ until $c(t) = 0$ to get perfect demodulation.

3. (a) From the Fourier transform integral and the Euler identity $2\cos(a) = e^{ja} + e^{-ja}$,

$$\begin{aligned}
&\mathcal{F}\big\{m(t)\cos(2\pi f_c t + \theta)\big\} \\
&= \int_{-\infty}^{\infty} m(t)\cos(2\pi f_c t + \theta) e^{-j2\pi f t} dt && (9) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} m(t)\big[e^{j2\pi f_c t + j\theta} + e^{-j2\pi f_c t - j\theta}\big] e^{-j2\pi f t} dt && (10) \\
&= \frac{1}{2}e^{j\theta}\int_{-\infty}^{\infty} m(t) e^{-j2\pi(f-f_c)t} dt + \frac{1}{2}e^{-j\theta}\int_{-\infty}^{\infty} m(t) e^{-j2\pi(f+f_c)t} dt && (11) \\
&= \frac{1}{2}\Big[e^{j\theta}M(f-f_c) + e^{-j\theta}M(f+f_c)\Big] && (12)
\end{aligned}$$

(b) Because filtering by $C(f)$ corresponds to frequency-domain multiplication,

$$\begin{aligned}
S(f) &= C(f)\cdot\mathcal{F}\big\{m(t)\cos(2\pi f_c t + \theta)\big\} && (13) \\
&= \frac{1}{2}C(f)\Big[e^{j\theta}M(f-f_c) + e^{-j\theta}M(f+f_c)\Big] && (14)
\end{aligned}$$

(c) In the previous problem, we derived the Fourier transform of the signal $m(t)$ after multiplication by the cosine $\cos(2\pi f_c t + \theta)$ and subsequent filtering by $c(t)$. Here we want to essentially the same thing: derive the Fourier transform of the signal $s(t)$ after multiplication by the cosine $\cos(2\pi f_c t + \phi)$ and subsequent filtering by the LPF. Thus, repeating (13)-(14), we find

$$\begin{aligned}
V(f) &= \mathsf{LPF}(f)\cdot\mathcal{F}\big\{s(t)\,2\cos(2\pi f_c t + \phi)\big\} && (15) \\
&= \mathsf{LPF}(f)\Big[e^{j\phi}S(f-f_c) + e^{-j\phi}S(f+f_c)\Big] && (16) \\
&= \frac{1}{2}\mathsf{LPF}(f)\Big[e^{j(\phi+\theta)}C(f-f_c)M(f-2f_c) + e^{j(\phi-\theta)}C(f-f_c)M(f) \\
&\quad + e^{-j(\phi-\theta)}C(f+f_c)M(f) + e^{-j(\phi+\theta)}C(f+f_c)M(f+2f_c)\Big] && (17) \\
&= \frac{1}{2}M(f)\Big[e^{j(\phi-\theta)}C(f-f_c) + e^{-j(\phi-\theta)}C(f+f_c)\Big] && (18)
\end{aligned}$$

where lowpass filtering suppressed the $M(f-2f_c)$ and $M(f+2f_c)$ terms but preserved the $M(f)$ terms.

(d) Plugging $f = 0$ into the previous expression, and then using $C(f_c) = 1 = C(-f_c)$,

$$\begin{aligned}
V(0) &= \frac{1}{2}M(0)\Big[e^{j(\phi-\theta)}C(-f_c) + e^{-j(\phi-\theta)}C(f_c)\Big] && (19) \\
&= \frac{1}{2}M(0)\Big[e^{j(\phi-\theta)} + e^{-j(\phi-\theta)}\Big] && (20) \\
&= M(0)\cos(\phi - \theta) && (21)
\end{aligned}$$

(e) Under the perfect demodulation condition $\phi = \theta$, we have $\boxed{V(0) = M(0)}$. Thus, if we didn't know θ but we knew $M(0)$, then we could adjust ϕ until $V(0) = M(0)$ to get perfect demodulation.