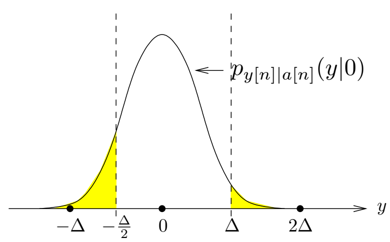
HOMEWORK SOLUTIONS #8

1. (a) Since $|a[n]|^2 = |-\Delta|^2 = \Delta^2$ with probability 1/3, $|a[n]|^2 = 0$ with probability 1/3, and $|a[n]|^2 = |2\Delta|^2 = 4\Delta^2$ with probability 1/3, we see that

$$\mathrm{E}\{|a[n]|^2\} = \sum_{a\in\{-\Delta,0,2\Delta\}} |a|^2 \cdot \Pr\{a[n]=a\} \tag{1}$$

$$= |-\Delta|^2\frac{1}{3} + |0|\frac{1}{3} + |2\Delta|^2\frac{1}{3} = \frac{5\Delta^2}{3}. \tag{2}$$

(b) To start, we draw the decision boundaries in the figure below, as well as the pdf of $y[n]$ conditioned on the event that $a[n]=0$.



Notice that the alphabet has a single interior point, $a=0$, and two exterior points, $a=-\Delta$ and $a=2\Delta$. Also notice that the decision boundaries are halfway between the elements of the alphabet.

Let's first consider the event that $a[n]=0$ (i.e., the interior point). Here, we make an error if $y[n] < -\frac{\Delta}{2}$ or $y[n] > \Delta$. The probability of $y[n] > \Delta$, given that $a[n]=0$, equals the integral of $p_{y[n]|a[n]}(y|0)$ over the region $y \in (\Delta, \infty)$, as illustrated by the right shaded area in the figure above. Since $y[n] = a[n] + e[n]$ conditioned on $a[n]=0$ reduces to $y[n]=e[n]$, the pdf $p_{y[n]|a[n]}(y|0)$ is that of a Gaussian random variable with zero-mean and variance σ_e^2:

$$\frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{y^2}{2\sigma_e^2}\right).$$

Thus

$$\Pr\left\{y[n] > \Delta \,\middle|\, a[n]=0\right\} = \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{y^2}{2\sigma_e^2}\right) dy = Q\left(\frac{\Delta}{\sigma_e}\right) \tag{3}$$

Similarly, the probability that $y[n] < \Delta/2$, given that $a[n]=0$, equals

$$\Pr\left\{y[n] < -\frac{\Delta}{2} \,\middle|\, a[n]=0\right\} = \int_{-\infty}^{-\Delta/2} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{y^2}{2\sigma_e^2}\right) dy \tag{4}$$

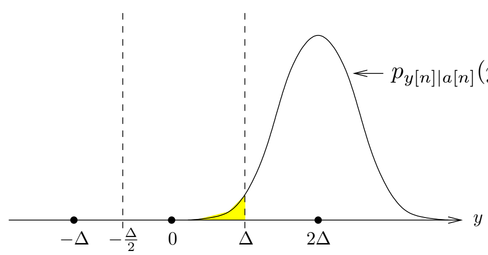
$$= \int_{\Delta/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{y^2}{2\sigma_e^2}\right) dy = Q\left(\frac{\Delta}{2\sigma_e}\right), \tag{5}$$

using the fact that the zero-mean Gaussian distribution is symmetric around the origin. Note that (5) was also derived in the lecture. Finally, the total probability of making a decision error, given that $a[n] = 0$, equals the probability of crossing the decision boundary to the left plus the probability of crossing the decision boundary to the right, i.e.,

$$\Pr\{\text{error}\big|a[n]=0\} \quad = \quad Q\left(\frac{\Delta}{\sigma_e}\right) + Q\left(\frac{\Delta}{2\sigma_e}\right). \tag{6}$$

Next, consider the event that $a[n] = 2\Delta$ (i.e., the right edge point). In this case, the pdf $p_{y[n]|a[n]}(y|2\Delta)$ is of interest, since (as in the figure below)

$$\Pr\{\text{error}\big|a[n]=2\Delta\} \quad = \quad \Pr\left\{y[n] < \Delta \middle| a[n] = 2\Delta\right\} \quad = \quad \int_{-\infty}^{\Delta} p_{y[n]|a[n]}(y|2\Delta)dy. \tag{7}$$



Because $y[n] = 2\Delta + e[n]$, we know $y[n]$ will be a Gaussian random variable with mean 2Δ and variance σ_e^2, and thus

$$p_{y[n]|a[n]}(y|2\Delta) \quad = \quad \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y-2\Delta)^2}{2\sigma_e^2}\right). \tag{8}$$

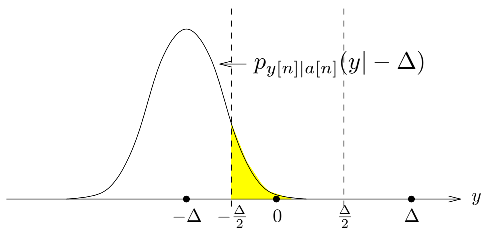
Plugging (8) into (7) yields

$$\begin{aligned}
\Pr\{\text{error}\big|a[n]=2\Delta\} &= \int_{-\infty}^{\Delta} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y-2\Delta)^2}{2\sigma_e^2}\right) dy & (9)\\
&= \int_{-\infty}^{-\Delta} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y')^2}{2\sigma_e^2}\right) dy' & (10)\\
&= \int_{\Delta}^{\infty} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y')^2}{2\sigma_e^2}\right) dy' & (11)\\
&= Q\left(\frac{\Delta}{\sigma_e}\right). & (12)
\end{aligned}$$

where (10) used the change of variables $y' = y - 2\Delta$, (11) used the symmetry property, and (12) took advantage of (3).

Finally, consider the event that $a[n] = -\Delta$ (i.e., the left edge point). Here the pdf $p_{y[n]|a[n]}(y|-\Delta)$ is of interest, since (as in the figure below)

$$\begin{aligned}
\Pr\{\text{error}\big|a[n]=-\Delta\} &= \Pr\left\{y[n] > -\frac{\Delta}{2} \middle| a[n] = -\Delta\right\} & (13)\\
&= \int_{-\Delta/2}^{\infty} p_{y[n]|a[n]}(y|-\Delta)dy. & (14)
\end{aligned}$$



Because $y[n] = -\Delta + e[n]$, we know $y[n]$ will be a Gaussian random variable with mean $-\Delta$ and variance σ_e^2, and thus

$$p_{y[n]|a[n]}(y|-\Delta) = \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y+\Delta)^2}{2\sigma_e^2}\right). \quad (15)$$

This means that

$$\begin{aligned} \Pr\{\text{error}\big|a[n] = -\Delta\} &= \int_{-\Delta/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y+\Delta)^2}{2\sigma_e^2}\right) dy & (16) \\ &= \int_{\Delta/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y')^2}{2\sigma_e^2}\right) dy' & (17) \\ &= Q\left(\frac{\Delta}{2\sigma_e}\right). & (18) \end{aligned}$$

where here we made the substitution $y' = y + \Delta$.

Putting (6), (12), and (18) together to get the average symbol-error probability:

$$\begin{aligned} \Pr\{\text{error}\} &= \Pr\{\text{error}\big|a[n] = -\Delta\} \Pr\{a[n] = -\Delta\} + \Pr\{\text{error}\big|a[n] = 0\} \Pr\{a[n] = 0\} \\ &\quad + \Pr\{\text{error}\big|a[n] = 2\Delta\} \Pr\{a[n] = 2\Delta\} & (19) \\ &= \frac{1}{3} Q\left(\frac{\Delta}{2\sigma_e}\right) + \frac{1}{3}\left[Q\left(\frac{\Delta}{\sigma_e}\right) + Q\left(\frac{\Delta}{2\sigma_e}\right)\right] + \frac{1}{3} Q\left(\frac{\Delta}{\sigma_e}\right) & (20) \\ &= \frac{2}{3}\left[Q\left(\frac{\Delta}{2\sigma_e}\right) + Q\left(\frac{\Delta}{\sigma_e}\right)\right]. & (21) \end{aligned}$$

(c) From (2) we see that

$$\Delta = \sqrt{\frac{3\,\mathrm{E}\{|a[n]|^2\}}{5}} \quad (22)$$

Plugging this into (21) gives

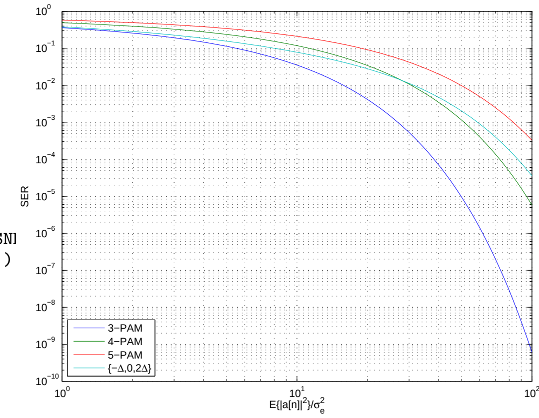
$$\Pr\{\text{error}\} = \frac{2}{3}\left[Q\left(\sqrt{\frac{3}{20}\frac{\mathrm{E}\{|a[n]|^2\}}{\sigma_e^2}}\right) + Q\left(\sqrt{\frac{3}{5}\frac{\mathrm{E}\{|a[n]|^2\}}{\sigma_e^2}}\right)\right]. \quad (23)$$

(d) The MATLAB code below yielded the following plot. Notice that the performance of the proposed alphabet is significantly worse than ordinary 3-PAM, and even worse than that of 4-PAM when $\mathrm{E}\{|a[n]|^2\}/\sigma_e^2 > 3$, though better than 5-PAM. This is disappointing! Here, “better” means lower SER for the same noise variance.

```
% setup
SNR = logspace(0,2,100);
M = [3,4,5];

% calculate SER
SER = zeros(length(M)+1,length(SNR));
legend_str = [];
for m=1:length(M),
 SER(m,:) = 2*(M(m)-1)/M(m)*Q(sqrt(3/(M(m)^2-1)*SNR));
 legend_str = strvcat(legend_str,[num2str(M(m)),'-PAM']);
end;
SER(length(M)+1,:) = 2/3*( Q(sqrt(3/20*SNR))+Q(sqrt(3/5*SN
legend_str = strvcat(legend_str,['\{-\Delta,0,2\Delta\}'])

% plot
handle = loglog(SNR,SER);
legend(handle,legend_str,'Location','SouthWest');
grid on;
ylabel('SER')
xlabel('E\{|a[n]|^2\}/\sigma_e^2')
```



Note: The code above calls the function `Q.m` which does $Q(x) = \mathrm{erfc}(x/\sqrt{2})/2$.

2. The MATLAB code below yielded the data in the following table.

	2-PAM	4-PAM	8-PAM
experimental SER	8.3400e-4	0.1181	0.4284
theoretical SER	7.8270e-4	0.1180	0.4289

Running the routine again yielded slightly different experimental SER values.

```
% generate symbols
N = 1e3; % # symbols
M = 2; % alphabet size
sig2a = 1; % symbol variance
a = pam(N,M,sig2s);  % symbol sequence
Delta = sqrt(12*sig2a/(M^2-1));

% generate error
sig2e = 0.1; % error variance
e = sqrt(sig2e)*randn(1,N);

% make decisions
y = a + e;
z = round( y*sqrt((M^2-1)/12) + (M-1)/2 );
z = min(max(z, 0), M-1);
ahat = (z - (M-1)/2)/sqrt((M^2-1)/12);

% count errors
err = zeros(1,N);
err(abs(a-ahat)>1e-10)=1;
SERhat = sum(err)/N
SER = 2*(M-1)/M*Q(sqrt(3/(M^2-1)*sig2s/sig2e))

if N>1000, return; end;

% visualize decision making
subplot(211)
plot([0:N-1],e,'.-',[0,N-1],Delta/2*[1,1],'r',[0,N-1],-Delta/2*[1,1],'r');
ylabel('e[n]')
subplot(212)
stem([0:N-1],err);
ylabel('errors')
```

3. To understand how the noise variance translates into symbol error variance, we recall the expression for the variance of the complex-valued error

$$E\{|e[n]|^2\} \quad = \quad \sigma_w^2 \|\underline{g}\|^2 + \sigma_a^2 \|\underline{H}^T \underline{q} - \underline{\delta}_D\|^2,$$

which we found while pursuing the MMSE equalizer. (Above, D is the end-to-end delay in symbol intervals.) The ISI term $\sigma_a^2 \|\underline{H}^T \underline{q} - \underline{\delta}_D\|^2$ is negligible in this problem because we are using well-truncated SRRC pulses with a trivial channel. Because $\|\underline{g}\|^2 \approx 1$, the noise term equals $\sigma_w^2 \|\underline{g}\|^2 = \sigma_w^2$. Finally, assuming that the real-valued part of $e[n]$ accounts for half of its power, we take $\sigma_e^2 = \frac{1}{2} E\{|e[n]|^2\} = \frac{1}{2}\sigma_w^2$ as the error variance that affects our decision making.

The MATLAB code below yielded the following constellation diagram, the experimental SER value 0.0252, and the theoretical SER value 0.0228. Running the routine again yielded a slightly different experimental SER value.

```
% design SRRC
P = 2; % oversampling factor
alpha = 0.5; % SRRC rolloff param
D = 2; % truncation to [-DT,DT]
g = srrc(D,alpha,P); % SRRC pulse
Ng = length(g);

% generate symbols
N = 1e4;  % # symbols
M = 2; % (sqrt) alphabet size
sig2a = 1; % symbol variance
a = pam(N,M,sig2a);  % symbol sequence

% pulse-amplitude modulate
a_up = zeros(1,N*P);
a_up(1:P:end) = a;  % upsampled symbols
m = conv(a_up,g); % PAM

% add complex-valued noise
sig2w = 0.5;
w = [1,j]*sqrt(sig2w/2)*randn(2,length(m));
v = m+w;

% matched-filter demodulate
q = g; Nq = Ng;
y_up = conv(v,q); % use SRRC again

% remove causal filtering delay
k = [1:P*N]; % desired time indices
dly = (Ng-1)/2+(Ng-1)/2;% delay due to pulses
y_up = y_up(k+dly); % remove delay
y = y_up(1:P:end); % downsample

% nearest-element decisions
z = round( real(y)*sqrt((M^2-1)/12) + (M-1)/2 );
z = min(max(z, 0), M-1);
ahat = (z - (M-1)/2)/sqrt((M^2-1)/12);

% count errors
err = zeros(1,N);
err(abs(a-ahat)>1e-10)=1;
SERhat = sum(err)/N

% compute theoretical error
H = convmtx(g,Nq);  H = H(:,[1:P:end]);  Na = size(H,2);
delta = zeros(1,Na); delta(1+dly/P)=1;
sig2e = sig2w/2*norm(q)^2 + sig2a*norm(q*H-delta)^2;
SER = 2*(M-1)/M*Q(sqrt(3/(M^2-1)*sig2a/sig2e))

% plot constellation diagram
if N>1e4, return; end;
figure(2)
plot(real(y_up),imag(y_up),'y',real(y),imag(y),'.');
xlabel('I'); ylabel('Q');
title(['SRRC (\alpha=',num2str(alpha),...
        ') truncated to \pm',num2str(D),'T'])
axis('equal');
```