HOMEWORK SOLUTIONS #6

1. From the left block diagram,

$$\begin{aligned} y(t) &= \big[x(t) * h_1(t)\big]e^{-j2\pi f_c t} && (1)\\ &= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau \cdot e^{-j2\pi f_c t} && (2)\\ &= \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f_c \tau}h_1(t-\tau)e^{-j2\pi f_c (t-\tau)}dt && (3)\end{aligned}$$

So, if $\boxed{h_2(t) = h_1(t)e^{-j2\pi f_c t}}$, we see that

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f_c \tau}h_2(t-\tau)dt && (4)\\ &= \big[x(t)e^{-j2\pi f_c t}\big] * h_2(t) && (5)\end{aligned}$$

which clearly describes the right block diagram.

2. Since $x(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$ is periodic with period T, it can be represented using a Fourier series with coefficients $X[k]$ as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\frac{2\pi}{T}t}, \qquad (6)$$

where

$$\begin{aligned} X[k] &:= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\frac{2\pi}{T}t}dt && (7)\\ &= \frac{1}{T}\int_{-T/2}^{T/2}\sum_{n=-\infty}^{\infty}\delta(t-nT)e^{-jk\frac{2\pi}{T}t}dt && (8)\\ &= \sum_{n=-\infty}^{\infty}\frac{1}{T}\int_{-T/2}^{T/2}\delta(t-nT)e^{-jk\frac{2\pi}{T}t}dt && (9)\\ &= \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\frac{2\pi}{T}t}dt \;=\; \frac{1}{T}, && (10)\end{aligned}$$

due to the sifting property of the Dirac delta. Thus we have

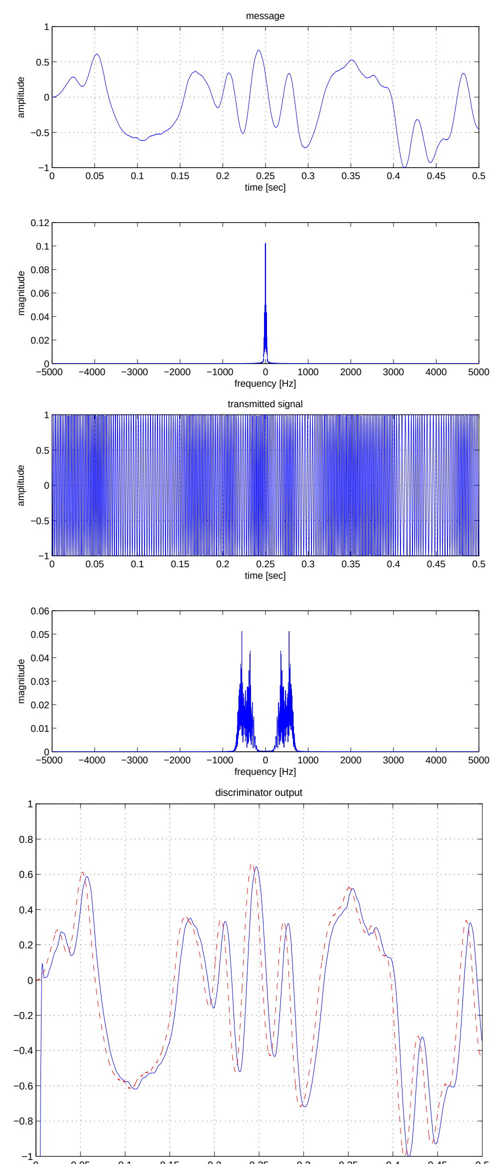
$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}. \qquad (11)$$

Taking the Fourier transform, we find

$$\begin{aligned}
\mathcal{F}\{x(t)\} &= \mathcal{F}\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}\right\} && (12)\\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathcal{F}\left\{e^{jk\frac{2\pi}{T}t}\right\} && (13)\\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(f-\frac{k}{T}). && (14)
\end{aligned}$$

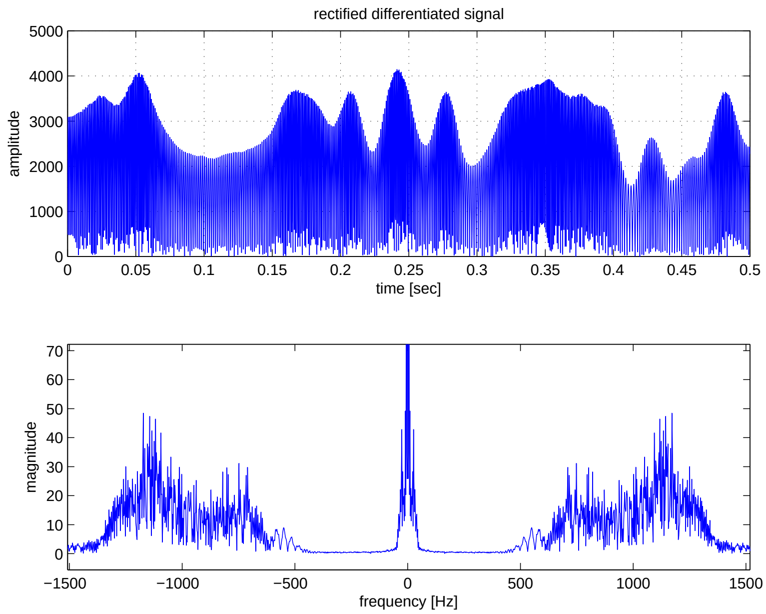
3. The matlab code for FM modulation and demodulation appears below. On the right are plots of the message signal, the transmitted signal, and the recovered signal.

```
randn('seed',0);

% generate message
Ts = 1/10000;
t_max = 0.5;
t = 0:Ts:t_max;
W = 50;
to = 20e-3;
h = firls(2*to/Ts,[0,0.25*W*2*Ts,W*2*Ts,1],[1,1,0,0])/Ts;
m = filter(h,1,randn(1,t_max/Ts+1))*Ts;
m = m/max(abs(m));          % make unit amplitude

figure(1);
 plottf(m,Ts);
 grid on; title('message');

% FM modulate
fc = 500;
D = 5;                      % modulation index
kf = W*D/max(abs(m));       % freq sensivity
BW99 = 2*(D+1)*W;           % Carson's BW
s = cos(2*pi*fc*t+2*pi*kf*cumsum(m)*Ts);

figure(2);
 plottf(s,Ts);
 grid on; title('transmitted signal');

% discriminator demodulate
dsdt = diff(s)/Ts;

figure(3);
 plottf(abs(dsdt),Ts);
 grid on; title('rectified differentiated signal');

tb = 5e-3;
b = firls(2*tb/Ts,[0,W*2*Ts,8*W*2*Ts,1],[1,1,0,0])/Ts;

figure(4);
 plottf(b,Ts);
 grid on; title('discriminator LPF');

env_dsdt = pi/2*filter(b,1,abs(dsdt))*Ts;
v = (env_dsdt-2*pi*fc)/(2*pi*kf);

figure(5);
 plot(t(1:end-1),v,t,m,'r--'); ylim([-1,1]);
 grid on; title('discriminator output');
```



The recovered signal looks quite close to the original message.

To design the discriminator LPF, it was helpful to look at the spectrum of the rectified differentiated signal. From the plot, one can see the message signal at baseband and corruption beginning at about 400 Hz.



Thus, a discriminator LPF was designed with passband edge at 50 Hz (i.e., the signal bandwidth) and stopband edge at 400 Hz (as shown below). The routine `firls` was used for this task, because it was found to perform slightly better than `firpm` and a lot better than `fir2`.