HOMEWORK SOLUTIONS #1

1. This problem can be solved using the inverse FT definition and Euler's representation of a cosine:

$$\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df & (1)\\
&= \int_{-W}^{0} e^{j\pi/2} e^{j2\pi f t} df + \int_{0}^{W} e^{-j\pi/2} e^{j2\pi f t} df & (2)\\
&= j\left[\frac{1}{j2\pi t} e^{j2\pi f t}\right]_{f=-W}^{0} - j\left[\frac{1}{j2\pi t} e^{j2\pi f t}\right]_{f=0}^{W} & (3)\\
&= \frac{1}{2\pi t}\left[1 - e^{-j2\pi W t}\right] - \frac{1}{2\pi t}\left[e^{j2\pi W t} - 1\right] & (4)\\
&= \frac{1}{\pi t} - \frac{1}{2\pi t}\left[e^{j2\pi W t} + e^{-j2\pi W t}\right] & (5)\\
&= \frac{1}{\pi t}\left[1 - \cos(2\pi W t)\right] & (6)
\end{aligned}$$

2. From p. 28 of the text, we know that $\mathcal{F}\{g^*(t)\} = G^*(-f)$. If you don't want to memorize this, it's easy to derive:

$$\mathcal{F}\{g^*(t)\} = \int g^*(t) e^{-j2\pi t f} dt = \left[\int g(t) e^{j2\pi t f} dt\right]^* = \left[\int g(t) e^{-j2\pi t(-f)} dt\right]^* = G^*(-f) \qquad (7)$$

Now, for a real-valued $g(t)$, we know that $g(t) = g^*(t)$, which implys that $G(f) = G^*(-f)$. Taking the absolute value, we see that that latter implies $|G(f)| = |G(-f)|$.

3. We would like to use

$$\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad G_1(f) G_2(f) \qquad (8)$$

to prove

$$\int_{-\infty}^{\infty} g_1(t) g_2^*(t-\tau) dt \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad G_1(f) G_2^*(f). \qquad (9)$$

One way to do this is to notice that if $g_2(t) = g_3^*(-t)$, then

$$G_2(f) = \int g_3^*(-t) e^{-j2\pi t f} dt = \left[\int g_3(-t) e^{j2\pi t f} dt\right]^* = \left[\int g_3(\tau) e^{-j2\pi \tau f} d\tau\right]^* = G_3^*(f). \qquad (10)$$

Applying $g_2(t) = g_3^*(-t)$ and (10) to (8), we get

$$\int_{-\infty}^{\infty} g_1(\tau) g_3^*(\tau - t) d\tau \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad G_1(f) G_3^*(f), \qquad (11)$$

which is equivalent to (9).

4. (a) To find $\mathcal{F}\{\cos^2(2\pi f_c t)\}$, we can apply the Euler identity $\cos(2\pi f_c t) = \frac{1}{2}\big[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\big]$ and then leverage $\mathcal{F}\{e^{j2\pi f_o t}\} = \delta(f - f_o)$ and the linearity property:

$$\begin{aligned} \cos^2(2\pi f_c t) &= \frac{1}{4}\big[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\big]^2 = \frac{1}{4}\big[e^{j2\pi 2 f_c t} + 2 + e^{-j2\pi 2 f_c t}\big] \quad (12) \\ \Rightarrow \mathcal{F}\{\cos^2(2\pi f_c t)\} &= \frac{1}{4}\delta(f - 2f_c) + \frac{1}{2}\delta(f) + \frac{1}{4}\delta(f + 2f_c) \quad (13) \end{aligned}$$

(b) To find $\mathcal{F}\{\sin^2(2\pi f_c t)\}$, we do the same thing with the Euler identity $\sin(2\pi f_c t) = \frac{1}{2j}\big[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\big]$

$$\begin{aligned} \sin^2(2\pi f_c t) &= -\frac{1}{4}\big[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\big]^2 = -\frac{1}{4}\big[e^{j2\pi 2 f_c t} - 2 + e^{-j2\pi 2 f_c t}\big] \quad (14) \\ \Rightarrow \mathcal{F}\{\sin^2(2\pi f_c t)\} &= -\frac{1}{4}\delta(f - 2f_c) + \frac{1}{2}\delta(f) - \frac{1}{4}\delta(f + 2f_c) \quad (15) \end{aligned}$$