Random Signals and Noise (Ch.8):

When designing a comm system, it is impossible to know exactly what the signal and noise waveforms will be. But usually we know average characteristics such as energy distribution across frequency. It is exactly these statistics that are most often used for comm system design.

We will consider random waveforms known as "zero-mean wide-sense stationary random processes." Such an $x(t)$ is completely completely characterized by its *power spectral density* $S_x(f)$, or average signal power versus frequency f. The technical definition of PSD accounts for the fact that power is defined as energy per unit time:

$$S_x(f) := \lim_{T\to\infty} \frac{1}{2T} \mathrm{E}\left\{ \left| \int_{-T}^{T} x(t) e^{j2\pi f t} dt \right|^2 \right\},$$

Above, $\mathrm{E}\{\cdot\}$ denotes *expectation*, i.e., statistical average.

A common random waveform is "white noise." Saying that $w(t)$ is *white* is equivalent to saying that its PSD is flat:

$$S_w(f) = N_o \text{ for all } f.$$

Broadband noise is often modelled as white random noise.

A fundamentally important question is: **What does filtering do to the power spectrum of a signal?** The answer comes with the aid of the *autocorrelation function* $R_x(\tau)$, defined as

$$R_x(\tau) := \mathrm{E}\{x(t)x^*(t-\tau)\}$$

and having the important property

$$R_x(\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} S_x(f).$$

Note that, for white noise $w(t)$ with PSD N_o, we have

$$\begin{aligned} S_w(f) = N_o \quad &\Rightarrow \quad R_w(\tau) = N_o\delta(\tau) \\ &\Rightarrow \quad \mathrm{E}\{w(t)w^*(t-\tau)\} = N_o\delta(\tau). \end{aligned}$$

Here we see that $w(t_1)$ and $w(t_2)|_{t_2\neq t_1}$ are "uncorrelated" because $\mathrm{E}\{w(t_1)w^*(t_2)\} = 0$, and that $w(t)$ has average energy $\mathcal{E}_w = \mathrm{E}\{|w(t)|^2\} = \infty$, which may be surprising. We could have found the same via $\mathcal{E}_w = \int_{-\infty}^{\infty} S_w(f)df$.

Now say that white noise $w(t)$ is filtered with non-random $h(t)$, yielding output $y(t) = \int_{-\infty}^{\infty} w(q)h(t-q)dq$. We can find $S_y(f)$ by first finding $R_y(\tau)$ and then taking the FT. Recall:

$$R_y(\tau) = \mathrm{E}\{y(t)y^*(t-\tau)\}.$$

Plugging in the expression for $y(t)$, we find

$$\begin{aligned} R_y(\tau) &= \mathrm{E}\left\{\int_{-\infty}^{\infty} w(q_1)h(t-q_1)dq_1 \int_{-\infty}^{\infty} w^*(q_2)h^*(t-\tau-q_2)dq_2\right\} \\ &= \mathrm{E}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(q_1)w^*(q_2)\, h(t-q_1)h^*(t-\tau-q_2)dq_1dq_2\right\}. \end{aligned}$$

Because averaging is a linear operation, and because the $h(\cdot)$ terms are fixed (non-random) quantities, we can write

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{E}\left\{w(q_1)w^*(q_2)\right\} h(t-q_1)h^*(t-\tau-q_2)dq_1dq_2.$$

Since $w(t)$ is white,

$$\mathrm{E}\left\{w(q_1)w^*(q_2)\right\} = \mathrm{E}\left\{w(q_1)w^*\big(q_1-(q_1-q_2)\big)\right\} = N_o\delta(q_1-q_2),$$

which allows use of the sifting property on the inner integral:

$$\begin{aligned} R_y(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N_o\delta(q_1-q_2)h(t-q_1)h^*(t-\tau-q_2)dq_1dq_2 \\ &= N_o\int_{-\infty}^{\infty} h(t-q_2)h^*(t-\tau-q_2)dq_2 \\ &= N_o\int_{\infty}^{-\infty} h(q)h^*(q-\tau)(-dq) \quad \text{using } q := t-q_2 \\ &= N_o\int_{-\infty}^{\infty} h(q)h^*(q-\tau)dq. \end{aligned}$$

To summarize, the autocorrelation of filtered white noise is

$$R_y(\tau) = N_o\int_{-\infty}^{\infty} h(q)h^*(q-\tau)dq.$$

Having $R_y(\tau)$, the final step is finding $S_y(f) = \mathcal{F}\{R_y(\tau)\}$:

$$\begin{aligned} S_y(f) &= \int_{-\infty}^{\infty} R_y(\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty}\left[N_o\int_{-\infty}^{\infty} h(q)h^*(q-\tau)dq\right]e^{-j2\pi f\tau}d\tau \\ &= N_o\int_{-\infty}^{\infty} h(q)e^{-j2\pi fq}\left[\int_{-\infty}^{\infty} h^*(q-\tau)e^{j2\pi f(q-\tau)}d\tau\right]dq \\ &= N_o\int_{-\infty}^{\infty} h(q)e^{-j2\pi fq}\left[\int_{-\infty}^{\infty} h^*(t)e^{j2\pi ft}dt\right]dq \\ &= N_o\int_{-\infty}^{\infty} h(q)e^{-j2\pi fq}dq\left[\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt\right]^* \\ &= N_oH(f)H^*(f) \;=\; N_o|H(f)|^2 \end{aligned}$$

To summarize, the power spectrum of filtered white noise is

$$S_y(f) = N_o|H(f)|^2.$$

More generally, the power spectrum of a filtered random process $x(t)$ can be shown to be

$$S_y(f) = S_x(f)\,|H(f)|^2,$$

which is quite intuitive.