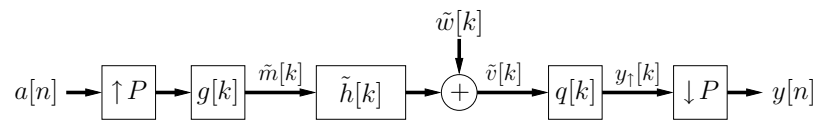


Error Analysis of Digital Comm (Ch.6,10):

When the channel is trivial and noiseless and the pulses satisfy the Nyquist criterion (i.e., $g[k] * q[k] = \delta[k]$), the digital comm system will work perfectly, yielding $y[n] = a[n]$.



In practice, however,

- the pulses $g[k]$ and $q[k]$ will be truncated to finite length,
- the channel will not be trivial (i.e., $\tilde{h}[k] \neq \delta[k]$), and
- the channel will not be noiseless (i.e., $\tilde{w}[k] \neq 0$),

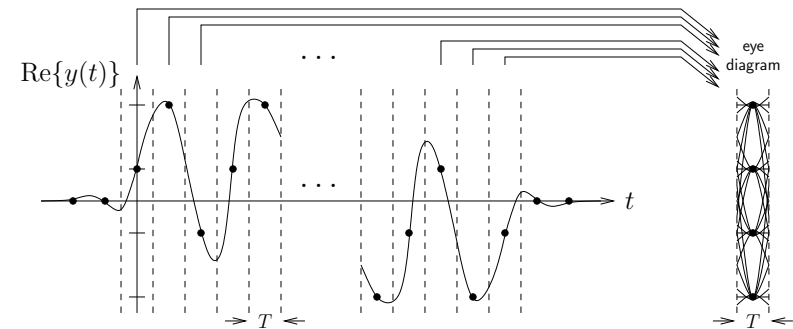
leading to $y[n] \neq a[n]$, in which case we must *infer* the value of $a[n]$ from the received samples $\{y[m]\}_{m=-\infty}^{\infty}$. For now, we consider using only the single sample $y[n]$ to infer $a[n]$.

Key question: What are the mechanisms by which errors are made?

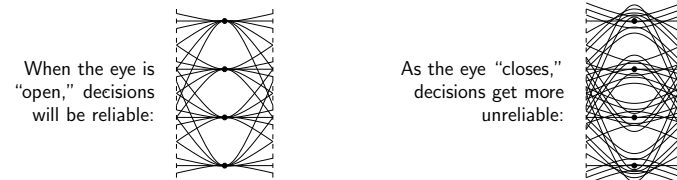
To better understand error behavior, we can plot the “eye diagram” or the “constellation diagram” and calculate the symbol error rate (SER).

Eye Diagrams:

Usually used when $a[n] \in \mathbb{R}$, the eye diagram is a plot which superimposes T -second segments of $\text{Re}\{y(t)\}$ over the time intervals $t \in [nT - \frac{T}{2}, nT + \frac{T}{2})$ for many integers n .



In MATLAB, the eye diagram can be made by superimposing P -sample segments of $\text{Re}\{y_{\uparrow}[k]\}$ corresponding to intervals $k \in \{nP - \frac{P}{2}, \dots, nP + \frac{P-1}{2}\}$ for many n . (Usually $P \geq 8$.)



When the eye is “open,” decisions will be reliable:

As the eye “closes,” decisions get more unreliable:

If $a[n] \in \mathbb{C}$, eye diagrams can be plotted for both the “I” and “Q” channels using $\text{Re}\{y(t)\}$ and $\text{Im}\{y(t)\}$, respectively.

MATLAB for digital mod/demod with eye diagram:

```

% design SRRC
P = 16; % oversampling factor
alpha = 0.5; % SRRC rolloff param
D = 2; % truncation to [-DT,DT]
g = srrc(D,alpha,P); % SRRC pulse
Ng = length(g);

% generate symbols
N = 100; % # symbols
M = 4; % alphabet size
sig2a = 1; % symbol variance
a = pam(N,M,sig2a); % symbol sequence

% pulse-amplitude modulate
a_up = zeros(1,N*P);
a_up(1:P:end) = a; % upsampled symbols
m = conv(a_up,g); % PAM

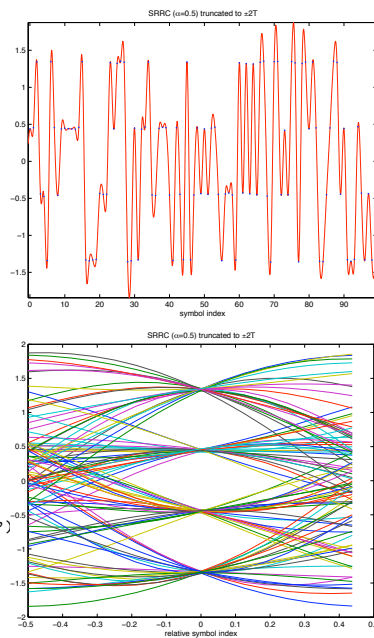
% matched-filter demodulate
y_up = conv(m,g); % use SRRC again

% remove causal filtering delay
left = -P/2; % left edge of eye diagram
k = left+[1:P*N]; % desired time indices
dly = (Ng-1)/2+(Ng-1)/2; % delay due to pulses
y_up = y_up(k+dly); % remove delay
y = y_up([1-left:P:P*N]); % downsample

% plot received signal
figure(1)
plot(left/P+[0:P*N-1]/P,y_up,'r',[0:N-1],y,',' )
axis('tight')
title(['SRRC (\alpha=',num2str(alpha),...
      ') truncated to \pm',num2str(D),'T'])
xlabel(['symbol index'])

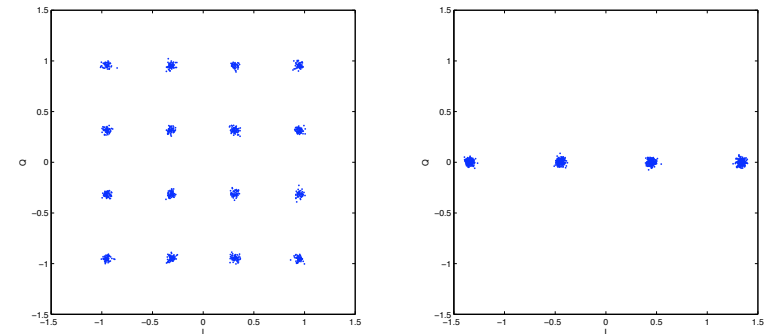
% plot eye diagram
figure(2)
Y_up = reshape(y_up,P,N); % extract segments
plot(left/P+[0:P-1]/P,Y_up) % superimpose them
title(['SRRC (\alpha=',num2str(alpha),...
      ') truncated to \pm',num2str(D),'T'])
xlabel(['relative symbol index'])

```



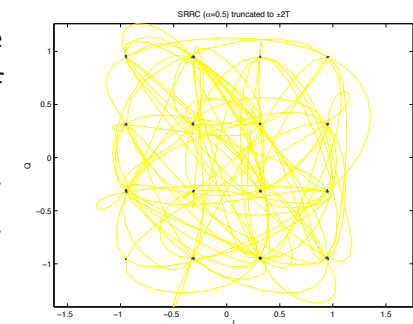
Constellation Diagrams:

The constellation diagram is a plot of $\text{Im}\{y[n]\}$ vs. $\text{Re}\{y[n]\}$ for many integers n . When the comm system is working well, the points cluster around the symbol alphabet values:



Recall that $y[n] \in \mathbb{C}$ due to the complex-baseband channel model, regardless of whether $a[n] \in \mathbb{R}$ or $a[n] \in \mathbb{C}$.

Sometimes it is instructive to superimpose a plot of $\text{Im}\{y_{\uparrow}[n]\}$ vs. $\text{Re}\{y_{\uparrow}[n]\}$, which approximates the trajectory of $y(t)$ in the complex plane:



MATLAB for digital mod/demod with constellation diagram:

```

% design SRRC
P = 8; % oversampling factor
alpha = 0.5; % SRRC rolloff param
D = 2; % truncation to [-DT,DT]
g = srrc(D,alpha,P); % SRRC pulse
Ng = length(g);

% generate symbols
N = 100; % # symbols
M = 4; % (sqrt) alphabet size
sig2a = 1; % symbol variance
a = qam(N,M,sig2a); % symbol sequence

% pulse-amplitude modulate
a_up = zeros(1,N*P);
a_up(1:P:end) = a; % upsampled symbols
m = conv(a_up,g); % PAM

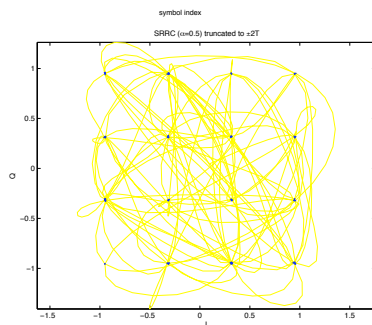
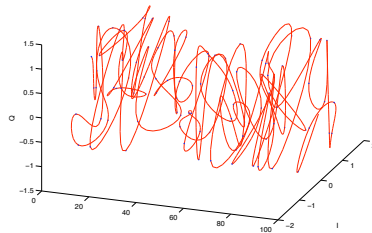
% matched-filter demodulate
y_up = conv(m,g); % use SRRC again

% remove causal filtering delay
k = [1:P*N]; % desired time indices
dly = (Ng-1)/2+(Ng-1)/2; % delay due to pulses
y_up = y_up(k+dly); % remove delay
y = y_up(1:P:end); % downsample

% plot received signal
figure(1)
plot3([0:P*N-1]/P,real(y_up),imag(y_up),...
'r',[0:N-1],real(y),imag(y),'.');
xlabel('symbol index');
ylabel('I'); zlabel('Q');
view(20,30); % to see full trajectory
%also try view(0,90), view(0,0), view(90,0)

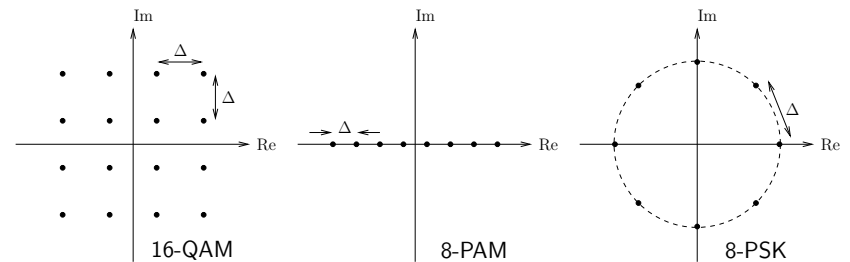
% plot constellation diagram
figure(2)
plot(real(y_up),imag(y_up),'y',real(y),imag(y),'.');
xlabel('I'); ylabel('Q');
title(['SRRC (\alpha=',num2str(alpha),...
' ) truncated to \pm',num2str(D),'T'])
axis('equal');

```



Popular Symbol Alphabets:

QAM : “quadrature amplitude modulation”
 PAM : “pulse amplitude modulation”
 PSK : “phase shift keying”



Note that “QPSK” = 4-QAM = 4-PSK
 “BPSK” = 2-PAM = 2-PSK

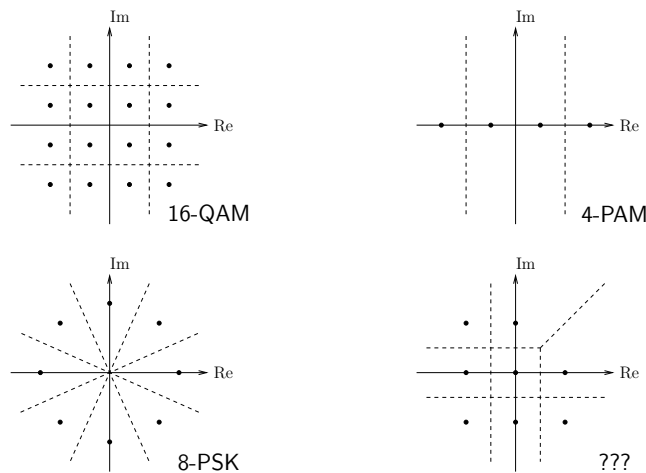
When the alphabet entries are spaced by Δ and picked with equal probability, the symbol variance $\sigma_a^2 = E\{|a[n]|^2\}$ obeys:

alphabet	M^2 -QAM	M -PAM	M -PSK
σ_a^2	$\frac{\Delta^2}{6}(M^2 - 1)$	$\frac{\Delta^2}{12}(M^2 - 1)$	$\frac{\Delta^2}{4 \sin^2(\pi/M)}$

Decision Regions:

A reasonable way to infer the transmitted symbol $a[n]$ from the received sample $y[n]$ is to decide that $a[n]$ was the alphabet element *nearest* to $y[n]$.

Nearest-element decision making is equivalent to using *decision regions* whose boundaries are equidistant from the two nearest alphabet elements:



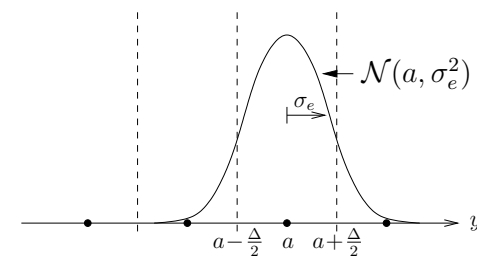
When $a[n] = a$, the *symbol error rate* (SER) equals the probability that $y[n]$ lies outside the decision region corresponding to alphabet member a .

Writing $y[n] = a[n] + e[n]$, we represent the cumulative effect of noise and ISI by the error $e[n]$. Usually we model $e[n]$ as a Gaussian random variable with mean 0 and variance σ_e^2 .

Symbol Error Rate (SER) for M -PAM:

Let's first consider an M -PAM alphabet, where $a[n] \in \mathbb{R}$. Since the decision regions show that $\text{Im}\{y[n]\}$ is not useful, we'll consider only the real parts of $y[n]$ and $e[n]$.

When $a[n] = a$, we have $y[n] = a + e[n]$, implying that $y[n]$ is Gaussian with mean a and variance σ_e^2 , abbreviated as " $\mathcal{N}(a, \sigma_e^2)$ ". This is illustrated below for the case of 4-PAM:



Formally, we say that $p_{y[n]|a[n]}(y|a)$, the probability density function (pdf) of $y[n]$ conditioned on $a[n] = a$, obeys

$$p_{y[n]|a[n]}(y|a) = \underbrace{\frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y-a)^2}{2\sigma_e^2}\right)}_{\mathcal{N}(a, \sigma_e^2)}.$$

Basically, $p_{y[n]|a[n]}(y|a)$ tells us how likely it is that $y[n] = y$ given that $s[n] = a$.

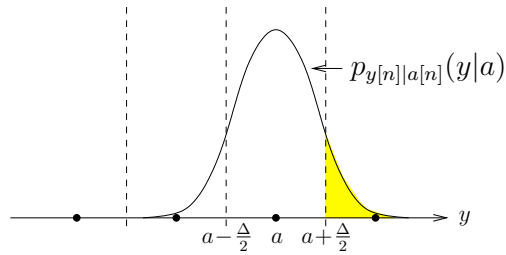
Consider first the case where a is an “interior” (not an “edge”) element of the symbol alphabet. Given that $a[n] = a$, we make an error when $y[n] < a - \frac{\Delta}{2}$ or when $y[n] > a + \frac{\Delta}{2}$. To find the probability of the latter error event, i.e.,

$$\Pr\{y[n] > a + \frac{\Delta}{2} \mid a[n] = a\},$$

we integrate $p_{y[n]|a[n]}(y|a)$ over $y \in (a + \frac{\Delta}{2}, \infty)$:

$$\int_{a+\frac{\Delta}{2}}^{\infty} \underbrace{p_{y[n]|a[n]}(y|a)}_{\mathcal{N}(a, \sigma_e^2)} dy = \int_{a+\frac{\Delta}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(-\frac{(y-a)^2}{2\sigma_e^2}\right) dy.$$

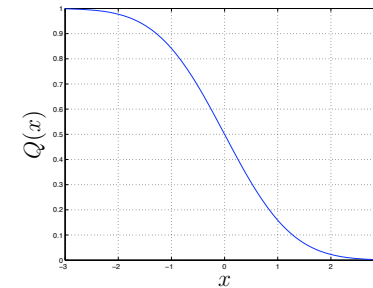
The integral represents the shaded area below:



This integral is often solved via

$$\int_x^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)}_{\mathcal{N}(\mu, \sigma^2)} dy = Q\left(\frac{x-\mu}{\sigma}\right),$$

using the “Q function”:



While the Q function is not represented in MATLAB, it can be calculated using the “complementary error function” erfc:

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

In any case, the latter error event occurs with probability

$$\begin{aligned} \Pr\{y[n] > a + \frac{\Delta}{2} \mid a[n] = a\} &= Q\left(\frac{(a + \frac{\Delta}{2}) - a}{\sigma_e}\right) \\ &= Q\left(\frac{\Delta}{2\sigma_e}\right). \end{aligned}$$

By symmetry, the former error event probability is also

$$\Pr\{y[n] < a - \frac{\Delta}{2} \mid a[n] = a\} = Q\left(\frac{\Delta}{2\sigma_e}\right).$$

Since these two events are disjoint, the probability of making a decision error on an interior symbol equals their sum:

$$Q\left(\frac{\Delta}{2\sigma_e}\right) + Q\left(\frac{\Delta}{2\sigma_e}\right) = 2Q\left(\frac{\Delta}{2\sigma_e}\right).$$

For edge symbols, we experience half the decision error probability, since there is only one decision boundary to cross.

Finally, we average over the conditional error probabilities:

$$\begin{aligned} \Pr\{\text{error}\} &= \sum_{a \in \text{alphabet}} \Pr\{\text{error} \mid a[n] = a\} \underbrace{\Pr\{a[n] = a\}}_{=1/M \text{ for all } a} \\ &= Q\left(\frac{\Delta}{2\sigma_e}\right) \cdot \frac{2}{M} + 2Q\left(\frac{\Delta}{2\sigma_e}\right) \cdot \frac{M-2}{M} \\ &= 2\left(\frac{M-1}{M}\right)Q\left(\frac{\Delta}{2\sigma_e}\right), \end{aligned}$$

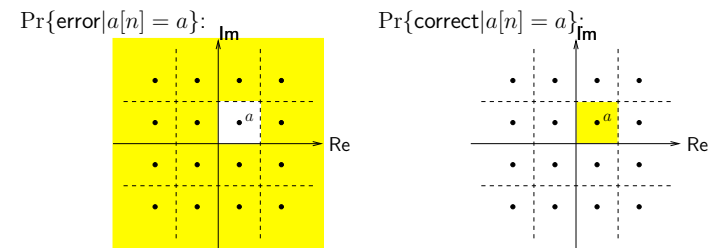
Using $\sigma_a^2 = \frac{\Delta^2}{12}(M^2 - 1)$, we can finally write

$$\text{SER}_{M\text{-PAM}} = 2\left(\frac{M-1}{M}\right)Q\left(\sqrt{\frac{3}{(M^2-1)}\frac{\sigma_a^2}{\sigma_e^2}}\right).$$

Symbol Error Rate (SER) for M^2 -QAM:

With QAM, we have complex-valued $y[n]$, $a[n]$, $e[n]$. We'll assume that $\text{Re}\{e[n]\}$ and $\text{Im}\{e[n]\}$ are uncorrelated and equal variance. To calculate SER, we can re-use the PAM approach with a few modifications:

1. integration is done on the complex plane,
2. σ_e^2 -variance $e[n] \Rightarrow \frac{\sigma_e^2}{2}$ -variance $\text{Re}\{e[n]\}$ & $\text{Im}\{e[n]\}$,
3. M^2 -QAM has 4 corner points, $4(M-2)$ edge points, and $M^2 - 4M + 4$ interior points,
4. calculate $\Pr\{\text{error} \mid a[n] = a\}$ via $1 - \Pr\{\text{correct} \mid a[n] = a\}$, since the regions of integration are simpler:



After a bit of algebra, we find

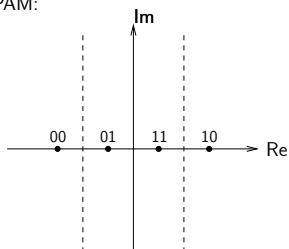
$$\text{SER}_{M^2\text{-QAM}} = 1 - \left[1 - 2\left(\frac{M-1}{M}\right)Q\left(\sqrt{\frac{3}{(M^2-1)}\frac{\sigma_a^2}{\sigma_e^2}}\right)\right]^2.$$

Bit Error Rate (BER) and Gray Coding:

With an M -ary alphabet, there are $\log_2 M$ bits per symbol, so a 1 symbol error could cause up to $\log_2 M$ bit errors.

Gray coding is a clever way of mapping bits to symbols so that neighboring symbols differ by only a single bit. Since the vast majority of errors occur when $y[n]$ falls into a neighboring decision region, Gray coding yields $\text{BER} \approx \text{SER}$.

4-PAM:



16-QAM:

