Derivation of M^2-QAM Symbol Error Rate

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For an interior M^2-QAM symbol, we have

$$\begin{aligned}
&\Pr\{\text{error} \mid a[n] = a_\mathsf{I} + ja_\mathsf{Q}\} \\
&= 1 - \Pr\{\text{correct} \mid a[n] = a_\mathsf{I} + ja_\mathsf{Q}\} \quad (1)\\
&= 1 - \int_{a_\mathsf{I}-\frac{\Delta}{2}}^{a_\mathsf{I}+\frac{\Delta}{2}} \int_{a_\mathsf{Q}-\frac{\Delta}{2}}^{a_\mathsf{I}+\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{(y_\mathsf{I}-a_\mathsf{I})^2}{2\frac{\sigma_e^2}{2}}\right) \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{(y_\mathsf{Q}-a_\mathsf{Q})^2}{2\frac{\sigma_e^2}{2}}\right) dy_\mathsf{Q} dy_\mathsf{I} \quad (2)\\
&= 1 - \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{(y_\mathsf{I}')^2}{2\frac{\sigma_e^2}{2}}\right) dy_\mathsf{Q}' \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{(y_\mathsf{I}')^2}{2\frac{\sigma_e^2}{2}}\right) dy_\mathsf{I}' \quad (3)\\
&= 1 - \left[\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{y^2}{2\frac{\sigma_e^2}{2}}\right) dy\right]^2. \quad (4)
\end{aligned}$$

We can recognize the integral as the probability of a correct decision for an interior PAM symbol when the error variance is $\sigma_e^2/2$. Thus,

$$\begin{aligned}
\Pr\{\text{error} \mid a[n] = a_\mathsf{I} + ja_\mathsf{Q}\} &= 1 - \left[1 - 2Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right]^2 \quad (5)\\
&= 4Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) - 4Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right). \quad (6)
\end{aligned}$$

For a M^2-QAM edge symbol, we have

$$\begin{aligned}
&\Pr\{\text{error} \mid a[n] = a_\mathsf{I} + ja_\mathsf{Q}\} \\
&= 1 - \Pr\{\text{correct} \mid a[n] = a_\mathsf{I} + ja_\mathsf{Q}\} \quad (7)\\
&= 1 - \int_{-\infty}^{\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{y_1^2}{2\frac{\sigma_e^2}{2}}\right) dy_1 \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{y_2^2}{2\frac{\sigma_e^2}{2}}\right) dy_2 \quad (8)
\end{aligned}$$

Here we recognize the first integral as the probability of correctly deciding a PAM edge symbol, and the second integral as the probability of correctly deciding a PAM interior symbol. Thus,

$$\begin{aligned}
\Pr\{\text{error} \mid a[n] = a_{\mathsf{I}} + j a_{\mathsf{Q}}\} &= 1 - \left[1 - Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right]\left[1 - 2Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right] &(9)\\
&= 3Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) - 2Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) &(10)
\end{aligned}$$

For a M^2-QAM corner symbol, we use the previous logic to obtain

$$\begin{aligned}
&\Pr\{\text{error} \mid a[n] = a_{\mathsf{I}} + j a_{\mathsf{Q}}\}\\
&\quad= 1 - \Pr\{\text{correct} \mid a[n] = a_{\mathsf{I}} + j a_{\mathsf{Q}}\} &(11)\\
&\quad= 1 - \int_{-\infty}^{\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{y_1^2}{2\frac{\sigma_e^2}{2}}\right) dy_1 \int_{-\infty}^{\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi\frac{\sigma_e^2}{2}}} \exp\left(-\frac{y_2^2}{2\frac{\sigma_e^2}{2}}\right) dy_2 &(12)\\
&\quad= 1 - \left[1 - Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right]^2 &(13)\\
&\quad= 2Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) - Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) &(14)
\end{aligned}$$

Since, for M^2-QAM, we have 4 corner points, $4(M-2)$ edge points, and $M^2 - 4M + 4$ interior points, the average error probability is

$$\begin{aligned}
\Pr\{\text{error}\} &= \frac{M^2 - 4M + 4}{M^2}\left[4Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) - 4Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right]\\
&\quad + \frac{4(M-2)}{M^2}\left[3Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) - 2Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right]\\
&\quad + \frac{4}{M^2}\left[2Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) - Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right] &(15)\\
&= \left[\frac{4M^2 - 16M + 16}{M^2} + \frac{12M - 24}{M^2} + \frac{8}{M^2}\right] Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\\
&\quad - \left[\frac{4M^2 - 16M + 16}{M^2} + \frac{8M - 16}{M^2} + \frac{4}{M^2}\right] Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) &(16)\\
&= 4\left(\frac{M-1}{M}\right) Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) - 4\left(\frac{M-1}{M}\right)^2 Q^2\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right) &(17)\\
&= 1 - \left[1 - 2\left(\frac{M-1}{M}\right) Q\left(\frac{\Delta}{\sqrt{2}\sigma_e}\right)\right]^2 &(18)
\end{aligned}$$

From $\sigma_s^2 = \frac{\Delta^2}{6}(M^2 - 1)$ we find $\Delta = \sqrt{\frac{6\sigma_s^2}{M^2-1}}$, from which

$$\Pr\{\text{error}\} = 1 - \left[1 - 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{3}{(M^2-1)}\frac{\sigma_s^2}{\sigma_e^2}}\right)\right]^2. \qquad (19)$$