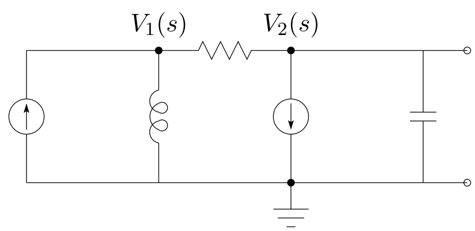
Node Voltage Analysis:

Summary:

1. Find initial conditions (ICs).
2. Transform circuit to s-domain using *current* sources for ICs. Assign reference and voltage nodes $\{V_1(s), V_2(s), \dots\}$.
3. Using KCL at each node, write circuit eqns in terms of $\{V_1(s), V_2(s), \dots\}$.
4. Solve eqns (using Cramer's rule, Matlab, etc.).



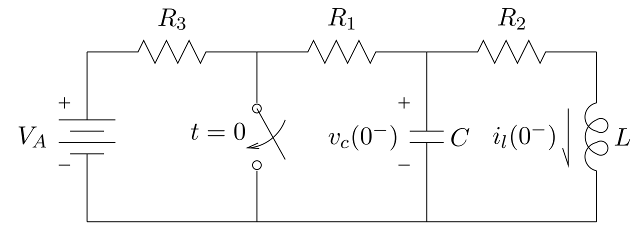
Notes:

a) May help to use admittances in step 3 above.

b) If there are voltage sources, then either ...

- Assign reference node at one end of the source.
- Convert to current source using Norton equivalent.
- Last resort: Define a supernode.

Example of Node Voltage Analysis:

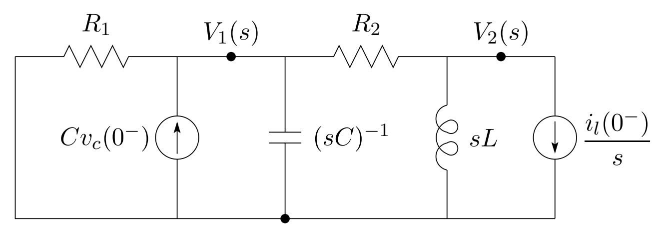


1) Find initial conditions: (open capacitor, short inductor)

$$i_l(0^-) = \frac{V_A}{R_1 + R_2 + R_3}$$

$$v_c(0^-) = R_2 i_l(0^-) = \frac{V_A R_2}{R_1 + R_2 + R_3}$$

2) Transform circuit to s-domain: (use currents for ICs)



3) Use KCL to set up circuit eqns:

$$\text{Node 1} : \frac{V_1(s)}{R_1} - Cv_c(0^-) + \frac{V_1(s)}{(sC)^{-1}} + \frac{V_1(s) - V_2(s)}{R_2} = 0$$

$$\text{Node 2} : \frac{V_2(s) - V_1(s)}{R_2} + \frac{V_2(s)}{sL} + \frac{i_l(0^-)}{s} = 0$$

Grouping like voltages...

$$\text{Node 1} \; : \; V_1(s)\left(\frac{1}{R_1} + sC + \frac{1}{R_2}\right) + V_2(s)\left(-\frac{1}{R_2}\right) \; = \; Cv_c(0^-)$$

$$\text{Node 2} \; : \; V_1(s)\left(-\frac{1}{R_2}\right) + V_2(s)\left(\frac{1}{sL} + \frac{1}{R_2}\right) \; = \; -\frac{i_l(0^-)}{s}$$

Rewriting in terms of admittances...

$$\text{Node 1} \; : \; V_1(s)\Big(G_1 + sC + G_2\Big) + V_2(s)\Big(-G_2\Big) \; = \; Cv_c(0^-)$$

$$\text{Node 2} \; : \; V_1(s)\Big(-G_2\Big) + V_2(s)\left(\frac{1}{sL} + G_2\right) \; = \; -\frac{i_l(0^-)}{s}$$

Matrix/vector representation:

$$\begin{bmatrix} sC + G_1 + G_2 & -G_2 \\ -G_2 & G_2 + \frac{1}{sL} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} Cv_c(0^-) \\ -\frac{i_l(0^-)}{s} \end{bmatrix}$$

4) To solve by hand, use Cramer's rule:

$$V_1(s) = \frac{\Delta_1(s)}{\Delta(s)}, \quad V_2(s) = \frac{\Delta_2(s)}{\Delta(s)}$$

where

$$\Delta(s) = \begin{vmatrix} sC + G_1 + G_2 & -G_2 \\ -G_2 & G_2 + \frac{1}{sL} \end{vmatrix}$$

$$\Delta_1(s) = \begin{vmatrix} Cv_c(0^-) & -G_2 \\ -\frac{i_l(0^-)}{s} & G_2 + \frac{1}{sL} \end{vmatrix}, \quad \Delta_2(s) = \begin{vmatrix} sC + G_1 + G_2 & Cv_c(0^-) \\ -G_2 & -\frac{i_l(0^-)}{s} \end{vmatrix}$$

Calculating determinants...

$$\begin{aligned}
\Delta(s) &= \begin{vmatrix} sC + G_1 + G_2 & -G_2 \\ -G_2 & G_2 + \frac{1}{sL} \end{vmatrix} \\
&= \Big(sC + G_1 + G_2\Big)\Big(G_2 + \frac{1}{sL}\Big) - G_2^2 \\
&= sCG_2 + G_1G_2 + \frac{sC + G_1 + G_2}{sL} \\
&= \frac{s^2LCG_2 + s(LG_1G_2 + C) + G_2 + G_2}{sL}
\end{aligned}$$

$$\begin{aligned}
\Delta_1(s) &= \begin{vmatrix} Cv_c(0^-) & -G_2 \\ -\frac{i_l(0^-)}{s} & G_2 + \frac{1}{sL} \end{vmatrix} \\
&= C\Big(G_2 + \frac{1}{sL}\Big)v_c(0^-) - \frac{G_2 i_l(0^-)}{s}
\end{aligned}$$

$$V_1(s) = \frac{\Delta_1(s)}{\Delta(s)} = \frac{sLC(G_2 + \frac{1}{sL})v_c(0^-) - LG_2 i_l(0^-)}{s^2LCG_2 + s(C + LG_1G_2) + G_1 + G_2}$$

$$\begin{aligned}
\Delta_2(s) &= \begin{vmatrix} sC + G_1 + G_2 & Cv_c(0^-) \\ -G_2 & -\frac{i_l(0^-)}{s} \end{vmatrix} \\
&= -\frac{i_l(0^-)}{s}\Big(sC + G_1 + G_2\Big) + CG_2 v_c(0^-)
\end{aligned}$$

$$V_2(s) = \frac{\Delta_2(s)}{\Delta(s)} = \frac{sLCG_2 v_c(0^-) - L(sC + G_1 + G_2) i_l(0^-)}{s^2LCG_2 + s(C + LG_1G_2) + G_1 + G_2}$$

Plugging in ICs and simplifying...

$$i_l(0^-) = \frac{V_A}{R_1+R_2+R_3} = G_2 v_c(0^-)$$
$$v_c(0^-) = \frac{V_A R_2}{R_1+R_2+R_3} = \frac{i_l(0^-)}{G_2}$$

$$\begin{aligned} V_1(s) &= \frac{sLC(G_2+\frac{1}{sL})v_c(0^-) - LG_2 i_l(0^-)}{s^2LCG_2 + s(C+LG_1G_2) + (G_1+G_2)} \\ &= \frac{sv_c(0^-) + \frac{v_c(0^-)}{LG_2} - \frac{i_l(0^-)}{C}}{s^2 + s(\frac{1}{LG_2}+\frac{G_1}{C}) + (\frac{G_1}{LCG_2}+\frac{1}{LC})} \\ &= \frac{s+\frac{1}{LG_2}-\frac{G_2}{C}}{s^2 + s(\frac{1}{LG_2}+\frac{G_1}{C}) + (\frac{G_1}{LCG_2}+\frac{1}{LC})} v_c(0^-) \\ &= \frac{s+\frac{R_2}{L}-\frac{1}{R_2C}}{s^2 + s(\frac{R_2}{L}+\frac{1}{R_1C}) + (\frac{R_2}{LCR_1}+\frac{1}{LC})} \cdot \frac{V_A R_2}{R_1+R_2+R_3} \end{aligned}$$

$$\begin{aligned} V_2(s) &= \frac{sLCG_2 v_c(0^-) - L(sC+G_1+G_2) i_l(0^-)}{s^2LCG_2 + s(C+LG_1G_2) + (G_1+G_2)} \\ &= \frac{s(v_c(0^-) - \frac{i_l(0^-)}{G_2}) - \frac{1}{C}(\frac{G_1}{G_2}+1) i_l(0^-)}{s^2 + s(\frac{1}{LG_2}+\frac{G_1}{C}) + (\frac{G_1}{LCG_2}+\frac{1}{LC})} \\ &= \frac{-\frac{1}{C}(\frac{G_1}{G_2}+1)}{s^2 + s(\frac{1}{LG_2}+\frac{G_1}{C}) + (\frac{G_1}{LCG_2}+\frac{1}{LC})} i_l(0^-) \\ &= \frac{-\frac{1}{C}(\frac{R_2}{R_1}+1)}{s^2 + s(\frac{R_2}{L}+\frac{1}{R_1C}) + (\frac{R_2}{LCR_1}+\frac{1}{LC})} \cdot \frac{V_A}{R_1+R_2+R_3} \end{aligned}$$

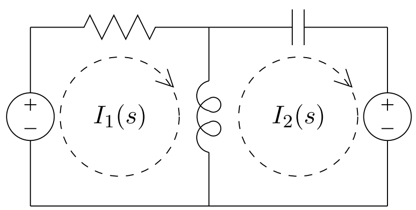
Did we make a mistake? Verify with Matlab...

```
>> clear all
>> syms s C L R1 R2 R3 V1 V2 VA iL vC
>> iL = VA/(R1 + R2 + R3);
>> vC = VA*R2/(R1 + R2 + R3);
>> syms tmp
>> tmp = inv( [s*C + 1/R1 + 1/R2, -1/R2; -1/R2, 1/R2 + 1/s/L] )...
*[C*vC; -iL/s];
>> pretty(tmp)

             [  (s L + R2) R1 C VA R2         R1 L VA        ]
             [  -------------------- - -----------------    ]
             [    %1 (R1 + R2 + R3)     %1 (R1 + R2 + R3)   ]
             [                                                ]
             [ R1 s L C VA R2     (s C R1 R2 + R2 + R1) L VA]
             [----------------- - -------------------------]
             [%1 (R1 + R2 + R3)      %1 (R1 + R2 + R3)     ]

                         2
                 %1 := s  C R1 L + s C R1 R2 + s L + R2 + R1
>> pretty( simplify(tmp) )

        [                                   2                  ]
        [              R1 VA (C R2 s L + C R2  - L)            ]
        [ ---------------------------------------------------- ]
        [   2                                                  ]
        [ (s  C R1 L + s C R1 R2 + s L + R2 + R1) (R1 + R2 + R3) ]
        [                                                      ]
        [                      L VA (R1 + R2)                  ]
        [- ----------------------------------------------------]
        [    2                                                 ]
        [  (s  C R1 L + s C R1 R2 + s L + R2 + R1) (R1 + R2 + R3)]
```

Mesh Current Analysis:

Summary:

1. Find initial conditions (ICs).
2. Transform circuit to s-domain using *voltage* sources for ICs. Assign mesh currents $\{I_1(s), I_2(s), \dots\}$.
3. Using KVL around each loop, write circuit eqns in terms of $\{I_1(s), I_2(s), \dots\}$.
4. Solve eqns (using Cramer's rule, Matlab, etc.).



Notes:

a) Applicable only if the circuit can be drawn on a flat surface without crossovers.

b) If there are current sources, then either ...

- Isolate current source in a single mesh current.
- Convert to voltage source using Thévenin equivalent.
- Last resort: Define a supermesh.

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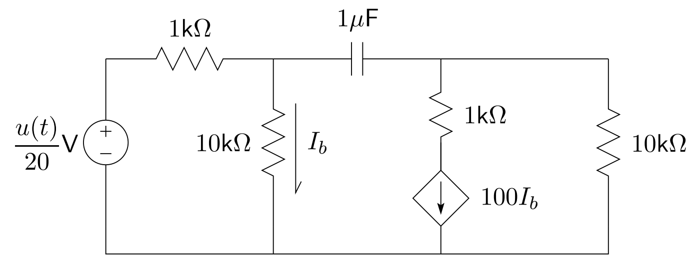
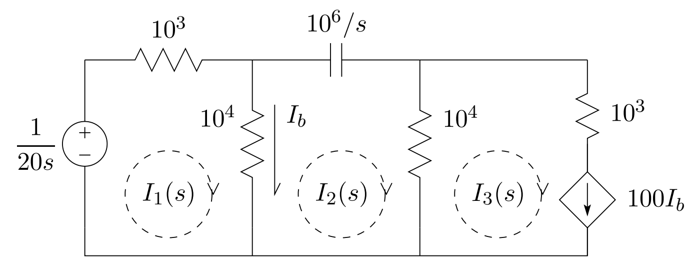
Example of Mesh Current Analysis:

Say no energy stored in capacitor for $t < 0$:



1) Initial currents and voltages equal zero.

2) Transforming to s-domain...



Current source isolated by rearranging circuit:

$$I_3(s) = 100 I_b = 100(I_1(s) - I_2(s))$$

Now don't need to solve for I_3 since it's determined by I_1 and I_2.

3) Use KVL to set up circuit equations:

$$\text{Mesh 1} \;:\; 10^3 I_1(s) + 10^4\Big(I_1(s) - I_2(s)\Big) - \frac{1}{20s} \;=\; 0$$

$$\text{Mesh 2} \;:\; \frac{10^6}{s} I_2(s) + 10^4\Big(I_2(s) - I_3(s)\Big) + 10^4\Big(I_2(s) - I_1(s)\Big) \;=\; 0$$

Using $I_3(s) = 100(I_1(s) - I_2(s))$ and collecting like currents...

$$\text{Mesh 1} \;:\; I_1(s)\Big(10^3 + 10^4\Big) + I_2(s)\Big(-10^4\Big) \;=\; \frac{1}{20s}$$

$$\text{Mesh 2} \;:\; I_1(s)\Big(-10^6 - 10^4\Big) + I_2(s)\Big(\frac{10^6}{s} + 10^4 + 10^6 + 10^4\Big) \;=\; 0$$

Matrix/vector representation:

$$\begin{bmatrix} 1.1e4 & -1e4 \\ -1.01e6 & \frac{1e^6}{s} + 1.02e6 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.05}{s} \\ 0 \end{bmatrix}$$

Using Cramer's rule...

$$\Delta(s) = \begin{vmatrix} 1.1e4 & -1e4 \\ -1.01e6 & \frac{1e^6}{s} + 1.02e6 \end{vmatrix} = \frac{1.1e10}{s} + 1.12e9 = \frac{1.12e9(s+9.82)}{s}$$

$$\Delta_1(s) = \begin{vmatrix} \frac{0.05}{s} & -1e4 \\ 0 & \frac{1e^6}{s} + 1.02e6 \end{vmatrix} = \frac{5e4}{s^2} + \frac{5.1e4}{s} = \frac{5.1e4(s+0.98)}{s^2}$$

$$\Delta_2(s) = \begin{vmatrix} 1.1e4 & \frac{0.05}{s} \\ -1.01e6 & 0 \end{vmatrix} = -\frac{5.05e4}{s}$$

$$I_1(s) = \frac{\Delta_1(s)}{\Delta(s)} = \frac{(4.55e\text{–}5)(s+0.98)}{s(s+9.82)}$$

$$I_2(s) = \frac{\Delta_2(s)}{\Delta(s)} = -\frac{(4.50e\text{–}5)}{s+9.82}$$

Using these currents, we can find voltages in circuit...

Voltage from top to bottom of left 10kΩ resistor:

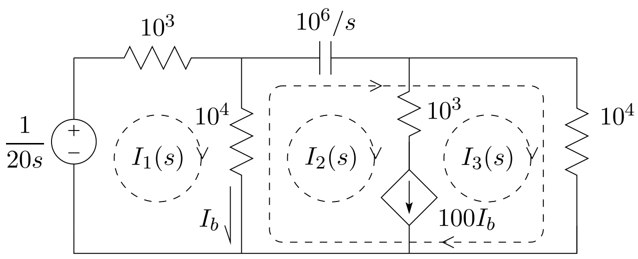
$$V_R(s) = \frac{I_1(s) - I_2(s)}{10^4} = \ldots$$

Voltage from left to right across capacitor:

$$V_C(s) = \frac{I_2(s)}{10^6 s^{-1}} = \ldots$$

What about the **supermesh** idea?

To see how this works, consider the same circuit but without the dependent source isolated in a single mesh:



The supermesh gives the equations

$$\begin{aligned} 0 &= 10^4\Big(I_2(s) - I_1(s)\Big) + \frac{10^6}{s} I_2(s) + 10^4 I_3(s) \\ I_2(s) - I_3(s) &= 100 I_b \;=\; 100\Big(I_1(s) - I_2(s)\Big) \end{aligned}$$

So if we plug the second equation into the first to eliminate $I_3(s)$...

$$\begin{aligned} 0 &= 10^4\Big(I_2(s) - I_1(s)\Big) + \frac{10^6}{s} I_2(s) + 10^4\Big(101 I_2(s) - 100 I_1(s)\Big) \\ &= I_1(s)\Big(-10^6 - 10^4\Big) + I_2(s)\Big(10^4 + \frac{10^6}{s} + 10^6 + 10^4\Big) \end{aligned}$$

which is exactly the “Mesh 2” equation we derived earlier!